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STABILITY OF THREE-DIMENSIONAL COMPRESSIBLE
BOUNDARY LAYERS

By Eli Reshotko

Lewis Research Center
Cleveland, Ohio

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SUMMARY

For Reynolds numbers sufficiently large that the dissipation terms in the disturbance energy equation are negligible, the stability of a three-dimensional boundary layer to a plane-wave disturbance of arbitrary orientation reduces to a two-dimensional stability problem governed by the boundary-layer velocity profile in the direction of wave propagation and by the mean temperature profile. Solution procedures are presented and the eigenvalue problem formulated including temperature fluctuations and a thermal boundary condition.

INTRODUCTION

The available studies of the stability of compressible laminar boundary layers to infinitesimal disturbances have been restricted essentially to two-dimensional boundary layers, while those few papers available on the stability of three-dimensional boundary layers consider only incompressible flow. With the current interest in compressible three-dimensional boundary layers encountered, for example, on swept wings or on yawed bodies in supersonic and hypersonic flow, it is of some interest to formulate an analysis for the stability of three-dimensional compressible boundary layers.

All the disturbances considered herein are plane waves. For a two-dimensional boundary layer those disturbances that propagate in the direction of boundary-layer development (local free-stream direction) are called two-dimensional disturbances, while those propagating at some angle to the local free-stream direction are called three-dimensional disturbances. For a three-dimensional boundary layer where the local free stream is not in the direction of the pressure gradient and, therefore, one might say that there is no single direction of boundary-layer

¹This work was begun while the author was at the California Institute of Technology and completed at the Lewis Research Center.

development, it is convenient to consider all disturbances as three-dimensional and to identify them by the angle of the direction in which they propagate relative to some reference direction.

The present study considers only "subsonic" disturbances, that is, disturbances that move subsonically with respect to the component of the free stream in the direction of wave propagation. Such disturbances have amplitudes that decay exponentially in the free stream. A disturbance that propagates supersonically with respect to the free stream would be expected to have a nonvanishing amplitude far from the wall.

The stability of two-dimensional compressible boundary layers to two-dimensional disturbances was first considered by Lees and Lin (refs. 1 and 2). They concluded that for subsonic and slightly supersonic flows the stability characteristics of a given boundary-layer profile are unaffected by temperature fluctuations. Specifically, the stability characteristics are determined entirely by satisfying velocity fluctuation boundary conditions. Dunn and Lin (ref. 3) found this conclusion to be invalid for moderately high supersonic Mach numbers, and they discussed thermal boundary conditions; however, they did not present any calculations that include consideration of the energy equation and thermal boundary conditions. This matter was further pursued by Dunn, Lin, and Mack and independently by Lees and the present author. The latter group, in a critical evaluation (refs. 4 and 5) of the order of magnitude and solution procedures used in solving the compressible boundary-layer stability problem, succeeded in identifying the leading and higher order viscous and conductive effects on the disturbances, and, in fact, used the leading and next order terms (including a dissipation term in the energy equation) in the calculation of neutral stability characteristics of insulated compressible boundary layers. The logical completion of the analysis of Dunn and Lin (ref. 3), including temperature fluctuations and a thermal boundary condition but considering only the leading viscous-conductive effects on the disturbances (e.g., no dissipation term in the energy equation), is presented in appendix D of reference 4 and also by Mack (ref. 6).

The studies of references 1 to 6 consider the boundary layer to be a "nearly-parallel" flow and treat the problem under the premise that the parallel flow disturbance equations apply; thus, terms involving the mean normal velocity and longitudinal derivatives of mean quantities are omitted. By means of the parallel flow assumptions the stability of a local profile is calculated as if only that profile existed from $-\infty$ to $+\infty$. In fact, Dunn (ref. 7) and Cheng (ref. 8) show that the mean vertical velocity does not enter until the second asymptotic approximation to the viscous solutions. Thus, if only the leading terms need be considered, the parallel flow approximation is a valid one. The work described herein makes the parallel flow assumptions.

With regard to the stability of two-dimensional parallel flows to three-dimensional disturbances, Squire (ref. 9) shows that for an incompressible fluid, the disturbance equations can be transformed to the completely two-dimensional Orr-Sommerfeld equation and that the two-dimensional disturbance is the least stable. Dunn and Lin (ref. 3) consider the stability of a two-dimensional compressible boundary layer to three-dimensional disturbances. They show that when only the leading viscous-conductive effects on the disturbances are considered the equations for three-dimensional disturbances can be transformed to those for two-dimensional disturbances. They carefully point out that for compressible flow these transformed equations are not the equations of a proper two-dimensional disturbance so that no "families of solutions" are obtainable; however, the transformation does permit the use of solution procedures for two-dimensional disturbances in problems of three-dimensional disturbances.

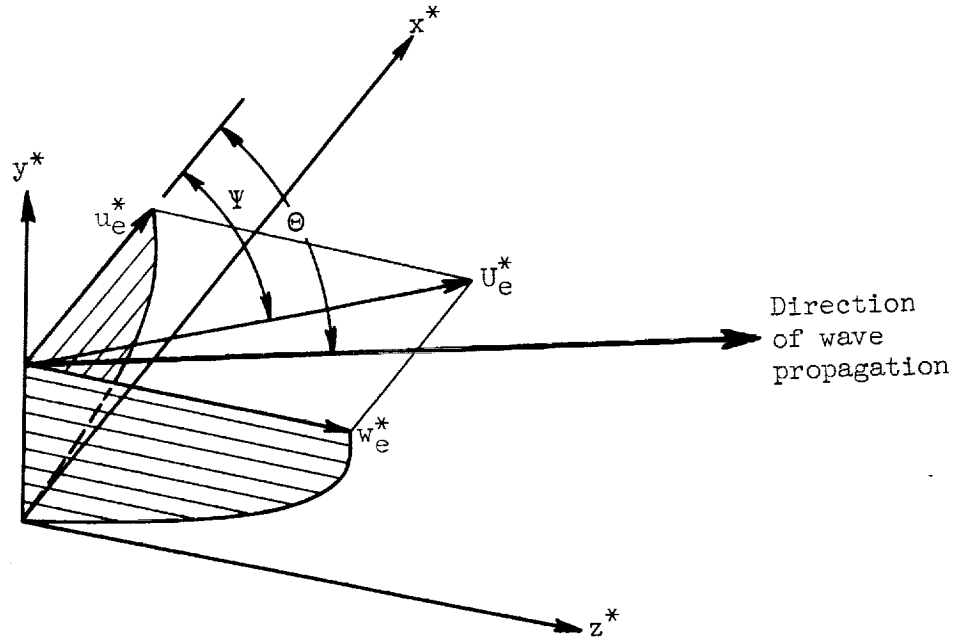
The stability of three-dimensional boundary layers to three-dimensional disturbances is considered for incompressible flow by Owen and Randall (ref. 10) and by Gregory, Stuart, and Walker (ref. 11). Their results for a parallel flow have been concisely summarized by Moore (ref. 12): "For a disturbance assumed to be moving in a certain direction, the eigenvalue problem may be treated as a two-dimensional one, governed by the boundary-layer velocity profile measured in that direction." Of course, for incompressible flow the energy equation is irrelevant and within the framework of the parallel flow assumption this statement is exact. It is shown herein that, for compressible flows, the transformation implied by Moore's statement applies exactly for the continuity and momentum equations but only for the leading terms of the energy equation. As already pointed out in reference 3, the dissipation terms do not all transform.

Accordingly, if consideration of the problem is restricted to the leading terms of the disturbance equations according to Dunn-Lin ordering, the eigenvalue problem may be treated as two-dimensional governed by the boundary-layer velocity profile in the direction of wave propagation and by the mean temperature profile. The formal solution of the problem, which is according to appendix D of reference 4, is presented and the resulting secular equation and a number of its special cases are discussed.

EQUATIONS OF INFINITESIMAL DISTURBANCES

Consider a point on the surface of a body on which there develops a three-dimensional boundary layer. It is assumed that the profile of the steady laminar boundary layer is known at this point in terms of the component profiles in two mutually orthogonal surface coordinate

directions x^* and z^* (see following sketch). The normal coordinate



is y^* . (All symbols are defined in appendix A.) The velocities in the x^* - and z^* -directions are u^* and w^* , respectively. The resultant external velocity $U_e^* = \sqrt{u_e^{*2} + w_e^{*2}}$ makes an angle $\Psi = \tan^{-1}(w_e^*/u_e^*)$ with the x^* -axis.

Dunn and Lin (ref. 3), by order of magnitude arguments, present disturbance equations for a parallel or nearly parallel flow at very large Reynolds number. The developments in the text of this report are based on the counterparts of the Dunn and Lin disturbance equations applicable to a three-dimensional mean flow. These equations are sufficient for the purposes of the present report; nevertheless, it is of some interest to go through the same operations with more complete disturbance equations so that the nature of the approximation will be better understood. Accordingly, the results for analogous operations on the complete parallel flow disturbance equations are given in appendix B. Taking the specific heats and Prandtl number to be constant and the viscosity to be a function of temperature alone, the disturbance equations for very large Reynolds numbers are:

Continuity:

$$\frac{\partial \rho^{*'}}{\partial t^*} + u^* \frac{\partial \rho^{*'}}{\partial x^*} + w^* \frac{\partial \rho^{*'}}{\partial z^*} + v^{*'} \frac{\partial \rho^*}{\partial y^*} = -\rho^* \left(\frac{\partial u^{*'}}{\partial x^*} + \frac{\partial v^{*'}}{\partial y^*} + \frac{\partial w^{*'}}{\partial z^*} \right) \quad (1)$$

Momentum:

$$\rho^* \left(\frac{\partial u^{*'}}{\partial t^*} + u^* \frac{\partial u^{*'}}{\partial x^*} + w^* \frac{\partial u^{*'}}{\partial z^*} + v^{*'} \frac{\partial u^{*'}}{\partial y^*} \right) = - \frac{\partial p^{*'}}{\partial x^*} + \mu^* \frac{\partial^2 u^{*'}}{\partial y^{*2}} \quad (2)$$

$$\rho^* \left(\frac{\partial v^{*'}}{\partial t^*} + u^* \frac{\partial v^{*'}}{\partial x^*} + w^* \frac{\partial v^{*'}}{\partial z^*} \right) = - \frac{\partial p^{*'}}{\partial y^*} + \mu^* \frac{\partial^2 v^{*'}}{\partial y^{*2}} \quad (3)$$

$$\rho^* \left(\frac{\partial w^{*'}}{\partial t^*} + u^* \frac{\partial w^{*'}}{\partial x^*} + w^* \frac{\partial w^{*'}}{\partial z^*} + v^{*'} \frac{\partial w^{*'}}{\partial y^*} \right) = - \frac{\partial p^{*'}}{\partial z^*} + \mu^* \frac{\partial^2 w^{*'}}{\partial y^{*2}} \quad (4)$$

Energy:

$$\begin{aligned} \rho^* \left(\frac{\partial T^{*'}}{\partial t^*} + u^* \frac{\partial T^{*'}}{\partial x^*} + w^* \frac{\partial T^{*'}}{\partial z^*} + v^{*'} \frac{\partial T^{*'}}{\partial y^*} \right) \\ = \frac{1}{c_p} \left(\frac{\partial p^{*'}}{\partial t^*} + u^* \frac{\partial p^{*'}}{\partial x^*} + w^* \frac{\partial p^{*'}}{\partial z^*} \right) + \frac{\mu^*}{\sigma} \frac{\partial^2 T^{*'}}{\partial y^{*2}} \end{aligned} \quad (5)$$

State:

$$\frac{p^{*'}}{p^*} = \frac{\rho^{*'}}{\rho^*} + \frac{T^{*'}}{T^*} \quad (6)$$

In these equations primed quantities are disturbance quantities, whereas unprimed quantities are mean quantities.

A significant characteristic of this system is the absence of dissipation terms in the energy equation. This absence makes possible the following transformation to a completely two-dimensional form. The dissipation terms are small only when the Reynolds number is very large so that the propriety of a given calculation depends on an a posteriori demonstration that the calculated Reynolds number is sufficiently large. It is shown in appendix B that the dissipation terms do not transform.

Dimensionless Variables

Expressing equations (1) to (6) in dimensionless form requires choosing suitable reference quantities. The reference length will be called L_{ref}^* and be of the order of magnitude of some boundary-layer thickness. The reference pressure is the mean boundary-layer pressure which is constant at the external stream value in accordance with the boundary-layer theory. For the boundary-layer mean flow, p is identically unity. The reference temperature will be some average temperature and will be called T_{ref}^* . Until the form of the disturbance

is specified, the choice of the reference velocity is not clear. Temporarily, it will be given the symbol U_{ref}^* . The reference time is then $L_{\text{ref}}^*/U_{\text{ref}}^*$. The dimensionless forms of equations (1) to (6) are:

Continuity:

$$\frac{\partial \rho'}{\partial t} + \bar{u} \frac{U_e^*}{U_{\text{ref}}^*} \cos \Psi \frac{\partial \rho'}{\partial x} + \bar{w} \frac{U_e^*}{U_{\text{ref}}^*} \sin \Psi \frac{\partial \rho'}{\partial z} + v' \frac{\partial \rho}{\partial y} = -\rho \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right) \quad (7)$$

Momentum:

$$\begin{aligned} \rho \left(\frac{\partial u'}{\partial t} + \bar{u} \frac{U_e^*}{U_{\text{ref}}^*} \cos \Psi \frac{\partial u'}{\partial x} + \bar{w} \frac{U_e^*}{U_{\text{ref}}^*} \sin \Psi \frac{\partial u'}{\partial z} + v' \frac{\partial \bar{u}}{\partial y} \frac{U_e^*}{U_{\text{ref}}^*} \cos \Psi \right) \\ = - \frac{1}{r M_{\text{ref}}^2} \frac{\partial p'}{\partial x} + \frac{1}{Re_{\text{ref}}} \mu \frac{\partial^2 u'}{\partial y^2} \end{aligned} \quad (8)$$

$$\begin{aligned} \rho \left(\frac{\partial v'}{\partial t} + \bar{u} \frac{U_e^*}{U_{\text{ref}}^*} \cos \Psi \frac{\partial v'}{\partial x} + \bar{w} \frac{U_e^*}{U_{\text{ref}}^*} \sin \Psi \frac{\partial v'}{\partial z} \right) \\ = - \frac{1}{r M_{\text{ref}}^2} \frac{\partial p'}{\partial y} + \frac{1}{Re_{\text{ref}}} \mu \frac{\partial^2 v'}{\partial y^2} \end{aligned} \quad (9)$$

$$\begin{aligned} \rho \left(\frac{\partial w'}{\partial t} + \bar{u} \frac{U_e^*}{U_{\text{ref}}^*} \cos \Psi \frac{\partial w'}{\partial x} + \bar{w} \frac{U_e^*}{U_{\text{ref}}^*} \sin \Psi \frac{\partial w'}{\partial z} + v' \frac{\partial \bar{w}}{\partial y} \frac{U_e^*}{U_{\text{ref}}^*} \sin \Psi \right) \\ = - \frac{1}{r M_{\text{ref}}^2} \frac{\partial p'}{\partial z} + \frac{1}{Re_{\text{ref}}} \mu \frac{\partial^2 w'}{\partial y^2} \end{aligned} \quad (10)$$

Energy:

$$\begin{aligned} \rho \left(\frac{\partial T'}{\partial t} + \bar{u} \frac{U_e^*}{U_{\text{ref}}^*} \cos \Psi \frac{\partial T'}{\partial x} + \bar{w} \frac{U_e^*}{U_{\text{ref}}^*} \sin \Psi \frac{\partial T'}{\partial z} + v' \frac{\partial T}{\partial y} \right) \\ = \left(\frac{\gamma - 1}{\gamma} \right) \left(\frac{\partial p'}{\partial t} + \bar{u} \frac{U_e^*}{U_{\text{ref}}^*} \cos \Psi \frac{\partial p'}{\partial x} + \bar{w} \frac{U_e^*}{U_{\text{ref}}^*} \sin \Psi \frac{\partial p'}{\partial z} \right) \\ + \frac{1}{Re_{\text{ref}}} \frac{\mu}{\sigma} \frac{\partial^2 T'}{\partial y^2} \end{aligned} \quad (11)$$

State:

$$p' = \frac{\rho'}{\rho} + \frac{T'}{T} \quad (12)$$

where

$$M_{\text{ref}} = \frac{U_{\text{ref}}^*}{\sqrt{\gamma R^* T_{\text{ref}}^*}} \quad (13)$$

$$Re_{\text{ref}} = \frac{U_{\text{ref}}^* L_{\text{ref}}^*}{\nu_{\text{ref}}^*} \quad (14)$$

Form of Disturbance

Consider the disturbance to be an oblique plane wave propagating at an angle Θ with respect to the x-direction. A fluctuating quantity is then described by the relation

$$Q'(x, y, z, t) = q(y) \exp[i\alpha(x \cos \Theta + z \sin \Theta - ct)] \quad (15)$$

where α is the wave number of the disturbance and c is the disturbance propagation velocity. The wave number is considered as a real quantity, while the propagation velocity is complex. Disturbances are neutral for $c_i = 0$ and are amplified for $c_i > 0$ or damped for $c_i < 0$.

Introduction of relations (15) into equations (7) to (12) yields the following set of reduced dimensionless disturbance equations: (From these equations on primes denote differentiation with respect to y .)

Continuity:

$$\begin{aligned} i \left(\bar{u} \frac{U_e^*}{U_{\text{ref}}^*} \cos \Theta \cos \Psi + \bar{w} \frac{U_e^*}{U_{\text{ref}}^*} \sin \Theta \sin \Psi - c \right) r + \rho' \varphi \\ = -\rho(\varphi' + i f \cos \Theta + i h \sin \Theta) \end{aligned} \quad (16)$$

Momentum:

$$\begin{aligned} \alpha \rho \left[i \left(\bar{u} \frac{U_e^*}{U_{\text{ref}}^*} \cos \Theta \cos \Psi + \bar{w} \frac{U_e^*}{U_{\text{ref}}^*} \sin \Theta \sin \Psi - c \right) f + \bar{\varphi} u' \frac{U_e^*}{U_{\text{ref}}^*} \cos \Psi \right] \\ = - \frac{i \alpha \pi \cos \Theta}{\gamma M_{\text{ref}}^2} + \frac{\mu}{Re_{\text{ref}}} f'' \end{aligned} \quad (17)$$

$$i\alpha^2\rho\left(\bar{u}\frac{U_e^*}{U_{ref}^*}\cos\Theta\cos\Psi+\bar{w}\frac{U_e^*}{U_{ref}^*}\sin\Theta\sin\Psi-c\right)\varphi=-\frac{\pi'}{rM_{ref}^2}+\frac{\mu}{Re_{ref}}\alpha\varphi'' \quad (18)$$

$$\begin{aligned} \alpha\rho\left[i\left(\bar{u}\frac{U_e^*}{U_{ref}^*}\cos\Theta\cos\Psi+\bar{w}\frac{U_e^*}{U_{ref}^*}\sin\Theta\sin\Psi-c\right)h+\varphi\bar{w}'\frac{U_e^*}{U_{ref}^*}\sin\Psi\right] \\ = -\frac{i\alpha\pi\sin\Theta}{rM_{ref}^2}+\frac{\mu}{Re_{ref}}h'' \end{aligned} \quad (19)$$

Energy:

$$\begin{aligned} \alpha\rho\left[i\left(\bar{u}\frac{U_e^*}{U_{ref}^*}\cos\Theta\cos\Psi+\bar{w}\frac{U_e^*}{U_{ref}^*}\sin\Theta\sin\Psi-c\right)\theta+\varphi T'\right] \\ = i\alpha\left(\frac{r-1}{r}\right)\left(\bar{u}\frac{U_e^*}{U_{ref}^*}\cos\Theta\cos\Psi+\bar{w}\frac{U_e^*}{U_{ref}^*}\sin\Theta\sin\Psi-c\right)\pi \\ + \frac{1}{Re_{ref}}\frac{\mu}{\sigma}\theta'' \end{aligned} \quad (20)$$

State:

$$\pi = \frac{r}{\rho} + \frac{\theta}{T} \quad (21)$$

Transformation to Two-Dimensional System

If an effective dimensionless mean velocity is defined as

$$W = \frac{U_e^*}{U_{ref}^*} (\bar{u}\cos\Theta\cos\Psi+\bar{w}\sin\Theta\sin\Psi) \quad (22)$$

and U_{ref}^* chosen so that $W = 1$ at the outer edge of the boundary layer, the reference velocity is

$$U_{ref}^* = U_e^* \cos(\Theta - \Psi) \quad (23)$$

As might have been expected, this quantity is the component of the resultant external velocity in the direction of wave propagation. Thus

$$W = \frac{\bar{u} \cos \Theta \cos \Psi + \bar{w} \sin \Theta \sin \Psi}{\cos(\Theta - \Psi)} \quad (24a)$$

$$= \frac{\bar{u} + \bar{w} \tan \Theta \tan \Psi}{1 + \tan \Theta \tan \Psi} \quad (24b)$$

When the x and z velocity disturbances are combined in the manner

$$\mathcal{F} = f \cos \Theta + h \sin \Theta \quad (25)$$

and the first momentum equation is formed from equation (17) multiplied by $\cos \Theta$ plus equation (19) multiplied by $\sin \Theta$, the disturbance equations become:

Continuity:

$$i(W - c)r + \rho' \varphi = -\rho(\varphi' + i \mathcal{F}) \quad (26)$$

Momentum:

$$\rho[i(W - c)\mathcal{F} + W'\varphi] = -\frac{i\pi}{\gamma M_{\text{ref}}^2} + \frac{\mu}{\alpha \text{Re}_{\text{ref}}} \mathcal{F}'' \quad (27)$$

$$i\rho\alpha^2(W - c)\varphi = -\frac{\pi'}{\gamma M_{\text{ref}}^2} + \frac{\mu}{\text{Re}_{\text{ref}}} \alpha\varphi'' \quad (28)$$

Energy:

$$\rho[i(W - c)\theta + T'\varphi] = i\left(\frac{\gamma - 1}{\gamma}\right)(W - c)\pi + \frac{1}{\alpha \text{Re}_{\text{ref}}} \frac{\mu}{\sigma} \theta'' \quad (29)$$

State:

$$\pi = \frac{r}{\rho} + \frac{\theta}{T} \quad (30)$$

The boundary conditions are:

At the wall, $y = 0$,

$$\left. \begin{array}{l} f = 0 \\ h = 0 \\ \varphi = 0 \\ \theta = 0 \end{array} \right\} \mathcal{J} = 0 \quad (31)$$

As $y \rightarrow \infty$,

$$\left. \begin{array}{l} f = 0 \\ h = 0 \\ \varphi = 0 \\ \theta = 0 \\ \pi = 0 \end{array} \right\} \mathcal{J} = 0 \quad (32)$$

The equations and boundary conditions (26) to (32) are in the form of those for a two-dimensional boundary layer with respect to two-dimensional disturbances. The techniques available from references 3 to 6 can be used in solving these equations. The wall-temperature boundary condition chosen here is not in the most general form. A more general boundary condition is $a\theta_w + b\theta'_w = 0$ where a and b are constants depending on the surface material, its thickness, the disturbance frequency and the method of internal cooling if one is employed (see appendix of ref. 3). Since most surface materials have thermal conductivities much greater than that of gases, any temperature fluctuations in the gas would be almost completely damped at the wall and the condition $\theta(0) = 0$ is reasonable for most purposes.

The solutions to equations (26) to (30) yield results for the combined longitudinal velocity fluctuation amplitude \mathcal{J} but not for the components f and h . To obtain f , it is necessary first to solve equations (26) to (30) and to determine the appropriate eigenvalues, then to substitute these eigenvalues and the corresponding φ and π eigenfunctions into equation (17). Integration of equation (17) will yield the distribution of the amplitude function f . The amplitude function h can then be obtained by using equation (25).

SOLUTION OF DISTURBANCE EQUATIONS

Following the procedures of references 1 to 7, two of the solutions to the linear system (eqs. (26) to (30)) are inviscid solutions, while the remaining ones are viscous.

Inviscid Solutions

A solution is sought to the disturbance equations of the form

$$q(y) = q_0(y) + \frac{1}{\alpha \tilde{Re}} q_1(y) + \dots \quad (33)$$

The resulting equations for the zeroth approximation q_0 are the inviscid equations, since they are the same as those obtained by ignoring viscosity and conductivity altogether. In the inviscid equations, which are as follows, the subscript zero is omitted since the q_0 functions are the only ones that will be obtained by this method:

Continuity:

$$\varphi' + i\mathcal{J} - \frac{T'}{T} \varphi + i(W - c) \left[\pi - \frac{\theta}{T} \right] = 0 \quad (34a)$$

Momentum:

$$i \left(\frac{W - c}{T} \right) \mathcal{J} + \frac{W'}{T} \varphi = - \frac{i\pi}{\gamma \tilde{M}^2} \quad (34b)$$

$$i\alpha^2 \left(\frac{W - c}{T} \right) \varphi = - \frac{\pi'}{\gamma \tilde{M}^2} \quad (34c)$$

Energy:

$$i \left(\frac{W - c}{T} \right) \theta + \frac{T'}{T} \varphi = i(W - c) \left(\frac{\gamma - 1}{\gamma} \right) \pi \quad (34d)$$

The Reynolds and Mach numbers in equations (33) and (34) that will be used for the remainder of the analysis are defined as

$$\left. \begin{aligned} \tilde{Re} &= \frac{U_{ref}^* L_{ref}^*}{\nu_e^*} \\ \tilde{M} &= \frac{U_{ref}^*}{\sqrt{\gamma R^* T_e^*}} \end{aligned} \right\} \quad (35)$$

Also, the mean and fluctuating temperatures will hereinafter be referred to the external mean flow temperature.

The quantities \mathcal{F} and θ can be eliminated from equations (34a), (34b), and (34d) so that equations (34) can be written (following ref. 1)

$$\left. \begin{aligned} \varphi' &= \frac{W'}{W - c} \varphi + \frac{i}{\gamma \tilde{M}^2} \left[\frac{T - \tilde{M}^2(W - c)^2}{W - c} \right] \pi \\ \pi' &= -i\alpha^2 \gamma \tilde{M}^2 \left(\frac{W - c}{T} \right) \varphi \end{aligned} \right\} \quad (36)$$

These equations can in turn be written as a second-order linear differential equation in either of the dependent variables φ or π . It has been customary to consider the solution of the second-order equation in the variable φ , which is proportional to the normal velocity fluctuation amplitude. In the present analysis, the solution procedure follows that of references 4 and 5 where the inviscid equation is written as follows in terms of the pressure fluctuation amplitude π :

$$\pi'' = \left(\frac{2W'}{W - c} - \frac{T'}{T} \right) \pi' + \alpha^2 \left[1 - \frac{\tilde{M}^2(W - c)^2}{T} \right] \pi \quad (37)$$

Following Heisenberg and Lin, it has been customary to solve the inviscid equation in the form of a convergent series in powers of α^2 . References 4 and 5 show that this procedure is inadequate for supersonic and hypersonic boundary layers and suggest the following more exact procedure. (A complete description of the Heisenberg-Lin procedure for compressible boundary layers is given by Mack (ref. 6).)

Equation (37) can be converted into a first-order nonlinear equation. Let

$$G = \frac{\pi'}{\alpha^2 \pi}$$

Equation (37) then becomes

$$G' = \left[1 - \frac{\tilde{M}^2(W - c)^2}{T} \right] + \left(\frac{2W'}{W - c} - \frac{T'}{T} \right) G - \alpha^2 G^2 \quad (38)$$

The outer boundary condition on G is obtained by considering the solutions to equation (37) for large y and for $1 - (1/\tilde{M}) \leq c \leq 1$. These solutions are

$$\pi \sim e^{\pm \alpha \sqrt{1 - \tilde{M}^2(1-c)^2} y} \quad (39)$$

Since disturbance amplitudes of subsonic disturbances are expected to decay far from the wall, the negative exponent is chosen in equation (39). The outer boundary condition on G is

$$G(\infty) = - \frac{\sqrt{1 - \tilde{M}^2(1 - c)^2}}{\alpha} \quad (40)$$

The balance of the procedure presented is for neutral disturbances, $c = c_r$, $c_i = 0$. Equation (38) is a complex equation and is split into the following real and imaginary equations:

$$G_r' = \left[1 - \frac{\tilde{M}^2(W - c)^2}{T} \right] + \left(\frac{2W'}{W - c} - \frac{T'}{T} \right) G_r - \alpha^2 (G_r^2 - G_i^2) \quad (41a)$$

$$G_i' = \left(\frac{2W'}{W - c} - \frac{T'}{T} \right) G_i - \alpha^2 (2G_r G_i) \quad (41b)$$

The outer boundary conditions are

$$\left. \begin{aligned} G_r(\infty) &= - \frac{\sqrt{1 - \tilde{M}^2(1 - c)^2}}{\alpha} \\ G_i(\infty) &= 0 \end{aligned} \right\} \quad (42)$$

There is a logarithmic singularity of the inviscid equation (eq. (37)) at the point where $W = c$; this point is referred to as the critical point. The expressions for G in the neighborhood of the critical point are obtained by series expansion (method of Frobenius) of equation (37) and the definition of G . The result from reference 4 is

$$\begin{aligned} G_r &= -\eta - A\eta^2 \ln|\eta| + (\text{const})\eta^2 - A^2\eta^3 \ln|\eta| \\ &+ \left[A(\text{const}) - \left(2B - 2A^2 + \frac{\tilde{M}^2 W_c'^2}{T_c} + \alpha^2 \right) \right] \eta^3 + \dots \end{aligned} \quad (43)$$

$$\left. \begin{aligned} \eta > 0: \\ \eta < 0: \end{aligned} \right\} G_i = \begin{cases} 0 \\ A\pi\eta^2(1 + A\eta + \dots) \end{cases} \quad (\pi = 3.14159) \quad (44)$$

where

$$\eta = y - y_c$$

$$A = \frac{W_c''}{W_c'} - \frac{T_c'}{T_c} = \frac{T_c}{W_c'} \left[\frac{d}{dy} \left(\frac{W'}{T} \right) \right]_c \quad (45)$$

and

$$B = \frac{W_c''^2}{4W_c'^2} + \frac{W_c'''}{3W_c'} - \frac{T_c''}{2T_c} - A \frac{T_c'}{T_c} \quad (46)$$

The calculation procedure for the inviscid solutions is as follows:
For a given profile (W,T) and value of c :

(1) Evaluate A and B . Choose a value of α . Evaluate the outer boundary condition from equation (42).

(2) Assume some value for the constant in equation (43). Evaluate G_r for a small positive value of η ($G_i = 0$).

(3) Continue evaluation of G_r by integration of equation (41a) to the outer edge of the mean flow. Compare the result with the outer boundary condition (42).

(4) Repeat steps 2 and 3, adjusting the constant until the outer boundary condition is satisfied.

(5) Use the value of the constant from step 4 to evaluate G_r and G_i for some small negative value of η .

(6) Continue evaluation of G_r and G_i by simultaneous integration of equations (41).

(7) Record at least the values of G_r and G_i at the wall ($y = 0$) since they will be used in determining the stability characteristics.

Viscous Solutions

The usual procedure in obtaining viscous solutions has been to solve a set of reduced equations that retain terms up to a certain order, near either the critical point or the surface. The ordering parameter is denoted ϵ where

$$\epsilon \sim \frac{1}{(\alpha Re)^n}$$

Lees and Lin (ref. 1) used an ordering procedure valid in the neighborhood of the critical layer to obtain viscous solutions, and then used these solutions to satisfy wall boundary conditions. Such a procedure, if at all valid, is so only when the critical point is close to the wall. The ordering procedure of Dunn and Lin (ref. 3) assumes the region of the wall to be distinct from the critical layer and leads to a set of reduced equations valid near the wall but not necessarily valid at the critical layer. Because of the desire to obtain proper representations of the rapidly varying viscous solutions in the vicinity of the wall to satisfy wall boundary conditions, it seems reasonable to use Dunn-Lin ordering at all times. The validity of Dunn-Lin ordering even in the limit $c \rightarrow 0$ now remains to be demonstrated.

The viscous effects near the wall are restricted to a narrow layer which does not include the critical point if it is shown that the thickness of that layer δ_w is smaller than the height to the critical point y_c . An approximate (asymptotic) magnitude for δ_w is

$$\delta_w \approx \left(\frac{2\nu_w}{\alpha \tilde{\text{Re}} c} \right)^{1/2}$$

while for $c \rightarrow 0$, $y_c \approx c/W_w'$ and thus

$$\frac{\delta_w}{y_c} \approx \frac{W_w'}{c} \left(\frac{2\nu_w}{\alpha \tilde{\text{Re}} c} \right)^{1/2}$$

This ratio may be more easily evaluated when expressed in terms of z , the argument of the Tietjens function. As $c \rightarrow 0$,

$$z^3 \approx \frac{\alpha \tilde{\text{Re}} c^3}{\nu_w W_w'^2}$$

so that the ratio δ_w/y_c may be estimated as

$$\frac{\delta_w}{y_c} \approx \frac{\sqrt{2}}{z^{3/2}}$$

As long as $z \gtrsim 2$, the premise underlying Lees-Lin ordering, namely, that the wall is in the neighborhood of the critical point, seems invalid. Since in most cases of interest $z \gtrsim 2$, it seems reasonable to expect Dunn-Lin ordering to hold for any surface temperature even as $c \rightarrow 0$. It is questionable whether Lees-Lin ordering ever applies at the wall. Strictly speaking, it probably does not apply, but practically speaking, it holds whenever the resulting equations are the same as by Dunn-Lin ordering which occurs, for example, for incompressible flow.

We now proceed with the Dunn-Lin ordering. Mean flow quantities and their derivatives are of unit order. With the quantity $(W - c)$ taken to be of order unity and differentiation of a disturbance amplitude with respect to y changing its order of magnitude by $1/\epsilon$, a consistent set of disturbance magnitudes are

$$r \sim \theta \sim \mathcal{F} \quad \varphi \sim \epsilon \mathcal{F} \quad \pi/\gamma M^2 \sim \epsilon^2 \mathcal{F} \quad (47)$$

If the quantity π is eliminated between equations (27) and (28) and the orders of magnitude of the various terms are evaluated with the aid of equation (47), the leading terms that result comprise the following equations:

Continuity:

$$\varphi' + i\mathcal{F} = \frac{i(W - c)\theta}{T} \quad (48)$$

Momentum:

$$\mathcal{F}''' - \frac{i\alpha \tilde{\text{Re}}(W - c)}{\nu} \mathcal{F}' = 0 \quad (49)$$

Energy:

$$\theta'' - \frac{i\alpha \tilde{\text{Re}}(W - c)}{\nu} \theta = 0 \quad (50)$$

with

$$\epsilon \sim \frac{1}{(\alpha \tilde{\text{Re}})^{1/2}} \quad (51)$$

Of the six linearly independent sets of solutions to equations (48) to (50), solutions 1 and 2 are identified with the inviscid solutions, solutions 3 and 4 are those where $\mathcal{F}' \neq 0$ but $\theta = 0$, and solutions 5 and 6 are those for which $\mathcal{F} = 0$ and $\theta \neq 0$. The aforementioned numbers will appear as subscripts to identify the solutions.

Consider first the momentum equation (eq. (49)). Following Tollmien (ref. 13), make the transformations

$$Y = \left[\int_{y_c}^y \frac{3}{2} \sqrt{\frac{W - c}{\nu}} dy \right]^{2/3} \quad (52)$$

and

$$\mathcal{F} = \mathcal{F}' \sqrt{\frac{dY}{dy}} = \mathcal{F}' \left(\frac{W - c}{vY} \right)^{1/4} \quad (53)$$

Then equation (49) becomes

$$\frac{d^2 \mathcal{F}}{dY^2} - \left[i\alpha \tilde{Re} Y + \mathcal{O}(1) \right] \mathcal{F} = 0 \quad (54)$$

The second term in the bracket of equation (54) is of order $1/\alpha \tilde{Re}$ compared with the first bracketed term and may be omitted. Thus equation (54) becomes

$$\frac{d^2 \mathcal{F}}{dY^2} - i\alpha \tilde{Re} Y \mathcal{F} = 0 \quad (55)$$

Now let

$$\xi = (\alpha \tilde{Re})^{1/3} Y \quad (56)$$

so that equation (55) becomes

$$\frac{d^2 \mathcal{F}}{d\xi^2} - i\xi \mathcal{F} = 0 \quad (57)$$

which is identical in form to that solved by Lees and Lin (ref. 1). Equation (57) has two linearly independent solutions

$$\mathcal{F}_3 = \xi^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\xi)^{3/2} \right] \quad (58a)$$

and

$$\mathcal{F}_4 = \xi^{1/2} H_{1/3}^{(2)} \left[\frac{2}{3} (i\xi)^{3/2} \right] \quad (58b)$$

Solution (58b) is rejected immediately since it grows exponentially for large ξ (large y) and cannot possibly satisfy the outer boundary conditions on \mathcal{F} and ϕ . From this set, therefore, only solutions with subscript 3 are pertinent to the problem. From equations (53) and (58a) the solution for \mathcal{F}_3 is

$$\mathcal{F}_3 = \left(\frac{dy}{dY} \right)^{3/2} \left(\frac{dY}{d\xi} \right) \int_{\infty}^{\xi} \xi^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\xi)^{3/2} \right] d\xi \quad (59a)$$

With recourse to equation (48) the solution for ϕ_3 is

$$\phi_3 = -i \left(\frac{dy}{dY} \right)^{5/2} \left(\frac{dY}{d\zeta} \right)^2 \int_{-\infty}^{\zeta} \int_{-\infty}^{\zeta} \zeta^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\zeta)^{3/2} \right] d\zeta d\zeta \quad (59b)$$

and, of course,

$$\theta_3 = 0 \quad (59c)$$

In performing the integrations to obtain \mathcal{F}_3 and ϕ_3 (and also ϕ_5 in eq. (65c)), quantities such as dy/dY , $dY/d\zeta$, and mean flow functions were taken outside the integral sign since they are slowly varying compared with the balance of the integrand. Slowly varying functions are those for which $d/dy \sim \mathcal{O}(1)$. This simplification incurs an error no larger than order $(\alpha \tilde{Re})^{-1/3}$.

The energy equation (eq. (50)) can be written in a form similar to equation (54) through the transformations

$$Y_0 = \left[\int_{Y_c}^Y \frac{3}{2} \sqrt{\frac{\sigma(W - c)}{v}} dy \right]^{2/3} \quad (60)$$

and

$$\Theta = \theta \sqrt{\frac{dY_0}{dy}} = \theta \left[\frac{\sigma(W - c)}{vY_0} \right]^{1/4} \quad (61)$$

After the unit order term is dropped, compared with the term of order $(\alpha \tilde{Re})$, let

$$\zeta_0 \equiv (\alpha \tilde{Re})^{1/3} Y_0 \quad (62)$$

Then the energy equation becomes

$$\frac{d^2 \Theta}{d\zeta_0^2} - i\zeta_0 \Theta = 0 \quad (63)$$

which has the solutions

$$\Theta_5 = \zeta_0^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\zeta_0)^{3/2} \right] \quad (64a)$$

and

$$\Theta_6 = \zeta_0^{1/2} H_{1/3}^{(2)} \left[\frac{2}{3} (i\zeta_0)^{3/2} \right] \quad (64b)$$

The solution Θ_6 grows exponentially for large ζ_0 (large y) so that it too is dropped because it cannot satisfy the outer boundary conditions. Then for solutions with subscript 5

$$\mathcal{F}_5 = 0 \quad (65a)$$

From equations (61) and (64a)

$$\theta_5 = \left(\frac{dy}{dY_0} \right)^{1/2} \zeta_0^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\zeta_0)^{3/2} \right] \quad (65b)$$

and from (65a), (65b), and the continuity equation (48)

$$\varphi_5 = \frac{i(W - c)}{T} \left(\frac{dy}{dY_0} \right)^{3/2} \left(\frac{dY_0}{d\zeta_0} \right) \int_{\infty}^{\zeta_0} \zeta_0^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\zeta_0)^{3/2} \right] d\zeta_0 \quad (65c)$$

Eigenvalue Problem

The solutions of the disturbance equations must now be combined to satisfy the boundary conditions. Note that the outer boundary conditions for subsonic disturbances (32) are inherently satisfied by choosing the negative exponent in equation (39) and by retaining only the viscous solutions 3 and 5, which decay exponentially far from the wall. The three boundary conditions at the wall (eqs. (31)) remain to be satisfied.

The inviscid functions will be denoted by the subscript *inv*. The satisfaction of boundary conditions (eqs. (31)) leads to the following determinantal relation:

$$\begin{vmatrix} \varphi_{\text{inv},w} & \varphi_{3,w} & \varphi_{5,w} \\ \mathcal{F}_{\text{inv},w} & \mathcal{F}_{3,w} & \mathcal{F}_{5,w} \\ \theta_{\text{inv},w} & \theta_{3,w} & \theta_{5,w} \end{vmatrix} = 0 \quad (66)$$

but from equations (59c) and (65a), $\theta_3 = \mathcal{F}_5 = 0$, so that the secular equation becomes

$$\frac{\varphi_{\text{inv},w}}{\mathcal{F}_{\text{inv},w}} = \frac{\varphi_{3,w}}{\mathcal{F}_{3,w}} + \frac{\theta_{\text{inv},w}}{\mathcal{F}_{\text{inv},w}} \frac{\varphi_{5,w}}{\theta_{5,w}} \quad (67)$$

With the aid of the identity derived from the inviscid equations (34b) and (34d),

$$\frac{\theta_{\text{inv},w}}{\mathcal{F}_{\text{inv},w}} = (\gamma - 1)\tilde{M}_c^2 + i(\gamma - 1)\tilde{M}_c^2 \left[\frac{W'_w}{c} - \frac{T'_w}{(\gamma - 1)\tilde{M}_c^2} \right] \frac{\varphi_{\text{inv},w}}{\mathcal{F}_{\text{inv},w}} \quad (68)$$

the secular equation (67) reduces to

$$\frac{\varphi_{\text{inv},w}}{\mathcal{F}_{\text{inv},w}} = \frac{\frac{\varphi_{3,w}}{\mathcal{F}_{3,w}} + (\gamma - 1)\tilde{M}_c^2 \frac{\varphi_{5,w}}{\theta_{5,w}}}{1 - i(\gamma - 1)\tilde{M}_c^2 \frac{\varphi_{5,w}}{\theta_{5,w}} \left[\frac{W'_w}{c} - \frac{T'_w}{(\gamma - 1)\tilde{M}_c^2} \right]} \quad (69)$$

Appearing in equation (69) are the ratios $\varphi_{3,w}/\mathcal{F}_{3,w}$ and $\varphi_{5,w}/\theta_{5,w}$, which upon evaluation from the viscous solutions are

$$\frac{\varphi_{3,w}}{\mathcal{F}_{3,w}} = -i \left(\frac{dy}{dY} \right)_w \left(\frac{dY}{d\xi} \right)_w \frac{\int_{-\infty}^{\xi_w} \int_{-\infty}^{\xi} \xi^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\xi)^{3/2} \right] d\xi d\xi}{\int_{-\infty}^{\xi_w} \xi^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\xi)^{3/2} \right] d\xi} \quad (70)$$

and

$$\frac{\varphi_{5,w}}{\theta_{5,w}} = - \frac{ic}{T_w} \left(\frac{dy}{dY_0} \right)_w \left(\frac{dY_0}{d\xi_0} \right)_w \frac{\int_{-\infty}^{\xi_{0w}} \xi_0^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\xi_0)^{3/2} \right] d\xi_0}{\int_{-\infty}^{\xi_{0w}} \xi_0^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\xi_0)^{3/2} \right] d\xi_0} \quad (71)$$

Let

$$\left. \begin{aligned} z &= -\xi_w \\ z_0 &= -\xi_{0w} \end{aligned} \right\} \quad (72)$$

also define the Tietjens function

$$F(z) \equiv \frac{- \int_{\infty}^{-z} \int_{\infty}^{\xi} \xi^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\xi)^{3/2} \right] d\xi \, d\xi}{z \int_{\infty}^{-z} \xi^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\xi)^{3/2} \right] d\xi} \quad (73)$$

and an auxiliary function

$$\tilde{G}(z_0) \equiv \frac{- \int_{\infty}^{-z_0} \xi_0^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\xi_0)^{3/2} \right] d\xi_0}{iz_0^{3/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i\xi_0)^{3/2} \right]} \quad (74)$$

Equations (70) and (71) can now be written

$$\frac{\Phi_{3,w}}{\mathcal{F}_{3,w}} = iz \left(\frac{dy}{dY} \right)_w \left(\frac{dY}{d\xi} \right)_w F(z) \quad (75)$$

and

$$\frac{\Phi_{5,w}}{\theta_{5,w}} = \frac{ic}{T_w} z_0 \left(\frac{dy}{dY} \right)_w \left(\frac{dY_0}{d\xi_0} \right)_w \tilde{G}(z_0) \quad (76)$$

Note here that $z_0 = \sigma^{1/3} z$ so that

$$z \left(\frac{dy}{dY} \right)_w \left(\frac{dY}{d\xi} \right)_w = z_0 \left(\frac{dy}{dY_0} \right)_w \left(\frac{dY_0}{d\xi_0} \right)_w$$

In terms of the universal functions $F(z)$ and $\tilde{G}(z_0)$, equation (69) becomes

$$\frac{\phi_{\text{inv},w}}{\mathcal{F}_{\text{inv},w}} = \frac{iz \left(\frac{dy}{dY}\right)_w \left(\frac{dY}{d\xi}\right)_w \left[F(z) + \frac{(\gamma - 1)\tilde{M}^2 c^2}{T_w} \tilde{G}(z_0) \right]}{1 + \frac{(\gamma - 1)\tilde{M}^2 c^2}{T_w} \tilde{G}(z_0) \left[\frac{W_w^1}{c} - \frac{T_w^1}{(\gamma - 1)\tilde{M}^2 c^2} \right] \left[z \left(\frac{dy}{dY}\right)_w \left(\frac{dY}{d\xi}\right)_w \right]} \quad (77)$$

Equation (77) is a proper secular equation; however, to avoid some of the difficulties involved in the expressions for $(dy/dY)_w (dY/d\xi)_w$, the equation is further reduced. Let

$$z \left(\frac{dy}{dY}\right)_w \left(\frac{dY}{d\xi}\right)_w \equiv (1 + \lambda) \frac{c}{W_w^1} \quad (78)$$

where it can be shown that

$$\lambda \equiv \frac{W_w^1}{c} \sqrt{\frac{v_w}{c}} \int_0^{y_c} \frac{3}{2} \sqrt{\frac{c - W}{v}} dy - 1 \quad (79)$$

Further, define

$$K \equiv \frac{(\gamma - 1)\tilde{M}^2 c^2}{T_w} \quad (80)$$

Since from the inviscid equations (34b) and (34c) and the definition $G \equiv (\pi'/\alpha^2 \pi)$,

$$\frac{\phi_{\text{inv},w}}{\mathcal{F}_{\text{inv},w}} = \frac{i}{\frac{W_w^1}{c} - \frac{1}{G_w}} \quad (81)$$

equation (77) can now be written, after some manipulation, as

$$-\frac{W_w^1}{c} G_w = \left[\frac{\Phi(z) - 1}{1 - \frac{\lambda}{\lambda + 1} \Phi(z)} \right] \frac{\left[1 + K \frac{\tilde{G}(z_0)}{F(z)} \right]}{\left[1 - \frac{\tilde{G}(z_0)\Phi(z)}{1 - \frac{\lambda}{\lambda + 1} \Phi(z)} \frac{T_w^1}{T_w} \frac{c}{W_w^1} \right]} \quad (82)$$

where

$$\Phi(z) \equiv \frac{1}{1 - F(z)} \quad (83)$$

is the modified Tietjens function. Manipulating the Tietjens function and its derivative with respect to z shows that

$$\tilde{G}(z) = \frac{F(z)}{1 - F(z) - zF'(z)} = \frac{\Phi(z) - 1}{1 - z \frac{\Phi'(z)}{\Phi(z)}} \quad (84)$$

The functions $F(z)$, $\tilde{G}(z)$, and $\Phi(z)$, as recently recomputed, are reproduced for convenience in table I. For $z > 10$ an asymptotic form of the secular equation (82) can be obtained as shown in appendix C. The result is

$$-\frac{W'_w}{c} G_w \approx (1 + \lambda)(1 + K\sigma^{-1/2})^p \left\{ 1 + i \left[1 + 2(1 + \lambda)^p \left(1 + \frac{T'_w c}{T_w W'_w} \sigma^{-1/2} \right) + \left(\frac{5 + 3K\sigma^{-1}}{2 + 2K\sigma^{-1/2}} \right)^p \right] + \dots \right\} \quad (85)$$

where

$$p = \frac{z^{-3/2}}{\sqrt{2}}, \quad z = \left(\frac{1}{p\sqrt{2}} \right)^{2/3} \quad (86)$$

Computation of Neutral Stability Characteristics

For a given boundary-layer profile (W, T) , choose a value of $c \geq 1 - (1/\tilde{M})$. Then

(1) Obtain the inviscid solution and record the values of G_w for various assumed values of α .

(2) Find the values of z and α for which equation (82) or (85) is satisfied.

(3) Compute $\alpha \tilde{Re}$ from the relation

$$\alpha \tilde{Re} = \frac{z^3}{\left(\frac{3}{2} \int_0^{y_c} \sqrt{\frac{c - W}{v}} dy \right)^2} \quad (87)$$

(4) The quantities α and $\alpha \tilde{Re}$ being known, the Reynolds number can be computed.

This procedure is to be repeated for successive values of c to obtain further points of neutral stability.

In some cases it may be convenient to use the Heisenberg-Lin formulation of the inviscid solution particularly if α is known to be a small quantity. A formulation for this case has been given by Mack (ref. 6). A slightly different formulation, which is related more closely to the present secular equation (82), is given in appendix D.

Special Cases of the Secular Equation

A few of the significant aspects of the current formulation are brought out through the following special cases:

Compressible boundary layers over insulated surfaces. - The secular equation for the stability of insulated boundary layers is obtained by taking $T_w' = 0$ in equation (82). The secular equation then becomes

$$-\frac{W_w'}{c} G_w = \left[\frac{\Phi(z) - 1}{1 - \frac{\lambda}{\lambda + 1} \Phi(z)} \right] \left[1 + K \frac{\tilde{G}(z_0)}{F(z)} \right] \quad (88)$$

The first bracketed term on the right side of equation (82) is a function of z alone, while the factor $K \equiv \frac{(\gamma - 1) \tilde{M}^2 c^2}{T_w}$ in the second

bracketed term introduces quantities dependent on the chosen boundary layer and value of c . These tend to make the eigenvalue determination more tedious. In the limit of Mach number zero ($\tilde{M} = 0$) only the first bracketed term survives and exactly the result of Lees and Lin (ref. 1) is obtained. Solutions of equation (88) have been obtained in references

4 and 6 where it is shown that the factor $K \frac{\tilde{G}(z_0)}{F(z)}$, which directly represents the effect of the temperature fluctuations, becomes important at Mach numbers of the order of 2. It is shown further in reference 4 that equation (88), which restricts consideration to the leading terms in the viscous solutions, gives adequate results only on the "upper branch" of the neutral curve where $\alpha \tilde{Re} \gg 1$. The next order terms in the viscous solutions, namely, the leading dissipation terms and higher order shear and conduction terms, have been considered and thoroughly discussed in references 4 and 5.

Mach number zero. - The secular equation for a Mach number zero flow is obtained by setting K equal to zero in equation (82). This procedure leaves

$$-\frac{W_w^t}{c} G_w = \frac{\left[\frac{\Phi(z) - 1}{1 - \frac{\lambda}{\lambda + 1} \Phi(z)} \right]}{\left[1 - \frac{\tilde{G}(z_0) \Phi(z)}{1 - \frac{\lambda}{\lambda + 1} \Phi(z)} \frac{T_w^t c}{T_w W_w^t} \right]} \quad (89)$$

The numerator on the right side of equation (89) represents the result of Lees and Lin (ref. 1), but for noninsulated surfaces, even at zero Mach number, the temperature fluctuations enter as shown by the denominator of equation (89).

CONCLUDING REMARKS

For Reynolds numbers sufficiently large that the dissipation terms in the disturbance energy equation are negligible, the stability of a three-dimensional boundary layer to a plane-wave disturbance of arbitrary orientation reduces to a two-dimensional stability problem governed by the boundary-layer velocity profile in the direction of wave propagation and by the mean temperature profile.

A procedure for exact calculation of the inviscid solution is presented while the viscous solutions including that for temperature fluctuations are obtained in terms of universal functions. The satisfaction of boundary conditions on velocity and temperature fluctuations leads to a secular equation from which it is apparent that consideration of the temperature fluctuations can have effect on the calculated eigenvalues for insulated compressible boundary layers as well as for non-insulated boundary layers at zero Mach number.

It is shown further that the viscous effects on the disturbance flow near the wall are restricted to a narrow layer, the thickness of which is always less than the distance to the critical layer, even in the case of the critical layer approaching the wall. This tends to support the applicability of Dunn-Lin ordering to all boundary-layer stability problems.

Lewis Research Center

National Aeronautics and Space Administration

Cleveland, Ohio, February 28, 1962

APPENDIX A

SYMBOLS

The symbols used in the present report are, in general, those commonly used in the literature on boundary-layer stability. In some regrettable instances a symbol will represent more than one item. The multiple uses will be indicated here and confusion should be minimized.

| | |
|--------------------|---|
| A | eq. (45) |
| B | eq. (46) |
| c | disturbance propagation velocity |
| c_p | specific heat at constant pressure |
| $F(z)$ | Tietjens function (eq. (73)) |
| $\mathcal{F}(y)$ | composite longitudinal velocity fluctuation amplitude, ($f \cos \Theta + h \sin \Theta$) |
| \mathcal{F} | $\mathcal{F}' \sqrt{dY/dy}$ |
| $f(y), h(y)$ | amplitudes of velocity fluctuations in x- and z-directions, respectively |
| G | $\pi' / \alpha^2 \pi$ |
| $\tilde{G}(z)$ | auxiliary function (eq. (74)) |
| K | $\frac{(\gamma - 1) \tilde{M}^2 c^2}{T_w}$ |
| k | thermal conductivity |
| L_{ref}^* | reference length |
| \tilde{M} | $U_{\text{ref}}^* / \sqrt{\gamma R^* T_e^*}$ |
| M_{ref} | $U_{\text{ref}}^* / \sqrt{\gamma R^* T_{\text{ref}}^*}$ |
| $m(y)$ | viscosity fluctuation amplitude |
| p | pressure |

| | |
|---------------|--|
| p | $z^{-3/2}/\sqrt{2}$ |
| $Q'(x,y,z,t)$ | fluctuating component of flow quantity |
| $q(y)$ | fluctuation amplitude |
| R^* | gas constant |
| \tilde{Re} | $U_{ref}^* L_{ref}^* / \nu_e^*$ |
| Re_{ref} | $U_{ref}^* L_{ref}^* / \nu_{ref}^*$ |
| $r(y)$ | density fluctuation amplitude |
| T | static temperature |
| T_e^* | local external temperature |
| T_{ref}^* | reference temperature |
| t | time |
| t_{ref}^* | L_{ref}^* / U_{ref}^* |
| U_e^* | resultant external velocity |
| U_{ref}^* | reference velocity, $U_e^* \cos(\Theta - \Psi)$ |
| u,v,w | velocities in x-, y-, and z-directions, respectively |
| u | real part of secular equation (appendix D) |
| \bar{u} | $u^* / u_e^* = u / u_e$ |
| v | imaginary part of secular equation (appendix D) |
| W | composite dimensionless mean velocity, $\frac{\bar{u} + \bar{w} \tan \Theta \tan \Psi}{1 + \tan \Theta \tan \Psi}$ |
| \bar{w} | $w^* / w_e^* = w / w_e$ |
| x,z | surface coordinate directions |
| Y | Tollmien variable for momentum equation |
| Y_0 | Tollmien variable for energy equation |

| | |
|--------------------|---|
| y | normal coordinate |
| z | $-\xi_w$ argument of the Tietjens function |
| z_0 | $-\xi_{0w}$ argument of the auxiliary function |
| α | wave number, $2\pi/\lambda$ |
| $\alpha\varphi(y)$ | normal velocity fluctuation amplitude |
| γ | ratio of specific heats |
| δ | boundary-layer thickness |
| δ_w | thickness of viscous layer near wall |
| ϵ | ordering parameter, $(\alpha\tilde{Re})^{-1/2}$ |
| ξ | $(\alpha\tilde{Re})^{1/3}Y$ |
| ξ_0 | $(\alpha\tilde{Re})^{1/3}Y_0$ |
| η | $y - y_c$ |
| Θ | angle of wave propagation direction relative to x-axis |
| $\theta(y)$ | temperature fluctuation amplitude |
| λ | wavelength |
| μ | absolute viscosity coefficient |
| μ_2 | second viscosity coefficient |
| ν | kinematic viscosity |
| $\pi(y)$ | pressure fluctuation amplitude |
| ρ | density |
| σ | Prandtl number |
| $\Phi(z)$ | modified Tietjens function |
| $\varphi(y)$ | quantity related to normal velocity fluctuation amplitude |
| Ψ | angle between resultant external velocity and x-axis |

Subscripts:

| | |
|-----|--|
| c | quantity evaluated at critical point |
| e | local condition outside mean boundary layer (external) |
| i | imaginary part |
| inv | inviscid |
| r | real part |
| ref | reference quantity |
| w | quantity evaluated at wall |

Superscripts:

| | |
|----|---|
| * | dimensional mean quantity |
| *' | dimensional fluctuating quantity |
| ' | denotes differentiation with respect to y beyond eq. (15) |

APPENDIX B

COMPLETE DISTURBANCE EQUATIONS FOR PARALLEL FLOW

The complete three-dimensional disturbance equations for a parallel flow with constant Prandtl number are as follows:

Continuity:

$$\frac{\partial \rho^{*'}}{\partial t^{*}} + u^{*} \frac{\partial \rho^{*'}}{\partial x^{*}} + w^{*} \frac{\partial \rho^{*'}}{\partial z^{*}} + v^{*'} \frac{\partial \rho^{*}}{\partial y^{*}} = -\rho^{*} \left(\frac{\partial u^{*'}}{\partial x^{*}} + \frac{\partial v^{*'}}{\partial y^{*}} + \frac{\partial w^{*'}}{\partial z^{*}} \right) \quad (B1)$$

Momentum:

$$\begin{aligned} \rho^{*} \left(\frac{\partial u^{*'}}{\partial t^{*}} + u^{*} \frac{\partial u^{*'}}{\partial x^{*}} + w^{*} \frac{\partial u^{*'}}{\partial z^{*}} + v^{*'} \frac{\partial u^{*}}{\partial y^{*}} \right) = & - \frac{\partial p^{*'}}{\partial x^{*}} \\ & + \mu^{*} \left(2 \frac{\partial^2 u^{*'}}{\partial x^{*2}} + \frac{\partial^2 u^{*'}}{\partial y^{*2}} + \frac{\partial^2 u^{*'}}{\partial z^{*2}} + \frac{\partial^2 v^{*'}}{\partial x^{*} \partial y^{*}} + \frac{\partial^2 w^{*'}}{\partial x^{*} \partial z^{*}} \right) \\ & + \frac{2}{3} (\mu_2^{*} - \mu^{*}) \left(\frac{\partial^2 u^{*'}}{\partial x^{*2}} + \frac{\partial^2 v^{*'}}{\partial x^{*} \partial y^{*}} + \frac{\partial^2 w^{*'}}{\partial x^{*} \partial z^{*}} \right) \\ & + \mu^{*'} \frac{\partial^2 u^{*}}{\partial y^{*2}} + \left(\frac{\partial u^{*}}{\partial y^{*}} \right) \left(\frac{\partial u^{*'}}{\partial y^{*}} \right) + \left(\frac{\partial u^{*}}{\partial y^{*}} \right) \left(\frac{\partial v^{*'}}{\partial x^{*}} + \frac{\partial u^{*'}}{\partial y^{*}} \right) \end{aligned} \quad (B2)$$

$$\begin{aligned} \rho^{*} \left(\frac{\partial v^{*'}}{\partial t^{*}} + u^{*} \frac{\partial v^{*'}}{\partial x^{*}} + w^{*} \frac{\partial v^{*'}}{\partial z^{*}} \right) = & - \frac{\partial p^{*'}}{\partial y^{*}} \\ & + \mu^{*} \left(\frac{\partial^2 v^{*'}}{\partial x^{*2}} + 2 \frac{\partial^2 v^{*'}}{\partial y^{*2}} + \frac{\partial^2 v^{*'}}{\partial z^{*2}} + \frac{\partial^2 u^{*'}}{\partial x^{*} \partial y^{*}} + \frac{\partial^2 w^{*'}}{\partial y^{*} \partial z^{*}} \right) \\ & + \frac{2}{3} (\mu_2^{*} - \mu^{*}) \left(\frac{\partial^2 u^{*'}}{\partial x^{*} \partial y^{*}} + \frac{\partial^2 v^{*'}}{\partial y^{*2}} + \frac{\partial^2 w^{*'}}{\partial y^{*} \partial z^{*}} \right) \\ & + 2 \left(\frac{\partial u^{*}}{\partial y^{*}} \right) \left(\frac{\partial v^{*'}}{\partial y^{*}} \right) + \frac{2}{3} \frac{\partial (\mu_2^{*} - \mu^{*})}{\partial y^{*}} \left(\frac{\partial u^{*'}}{\partial x^{*}} + \frac{\partial v^{*'}}{\partial y^{*}} + \frac{\partial w^{*'}}{\partial z^{*}} \right) \\ & + \left(\frac{\partial u^{*'}}{\partial x^{*}} \right) \left(\frac{\partial u^{*}}{\partial y^{*}} \right) + \left(\frac{\partial u^{*'}}{\partial z^{*}} \right) \left(\frac{\partial w^{*}}{\partial y^{*}} \right) \end{aligned} \quad (B3)$$

$$\begin{aligned}
\rho^* \left(\frac{\partial w^{*'}}{\partial t^*} + u^* \frac{\partial w^{*'}}{\partial x^*} + w^* \frac{\partial w^{*'}}{\partial z^*} + v^{*'} \frac{\partial w^*}{\partial y^*} \right) = & - \frac{\partial p^{*'}}{\partial z^*} \\
& + \mu^* \left(\frac{\partial^2 w^{*'}}{\partial x^{*2}} + \frac{\partial^2 w^{*'}}{\partial y^{*2}} + 2 \frac{\partial^2 w^{*'}}{\partial z^{*2}} + \frac{\partial^2 u^{*'}}{\partial x^* \partial z^*} + \frac{\partial^2 v^{*'}}{\partial y^* \partial z^*} \right) \\
& + \frac{2}{3} (\mu_2^* - \mu^*) \left(\frac{\partial^2 u^{*'}}{\partial x^* \partial z^*} + \frac{\partial^2 v^{*'}}{\partial y^* \partial z^*} + \frac{\partial^2 w^{*'}}{\partial z^{*2}} \right) \\
& + \mu^{*'} \frac{\partial^2 w^*}{\partial y^{*2}} + \left(\frac{\partial w^*}{\partial y^*} \right) \left(\frac{\partial \mu^{*'}}{\partial y^*} \right) + \left(\frac{\partial \mu^*}{\partial y^*} \right) \left(\frac{\partial w^{*'}}{\partial y^*} + \frac{\partial v^{*'}}{\partial z^*} \right) \quad (B4)
\end{aligned}$$

Energy:

$$\begin{aligned}
\rho^* \left(\frac{\partial T^{*'}}{\partial t^*} + u^* \frac{\partial T^{*'}}{\partial x^*} + w^* \frac{\partial T^{*'}}{\partial z^*} + v^{*'} \frac{\partial T^*}{\partial y^*} \right) = & \frac{1}{c_p} \left(\frac{\partial p^{*'}}{\partial t^*} + u^* \frac{\partial p^{*'}}{\partial x^*} + w^* \frac{\partial p^{*'}}{\partial z^*} \right) \\
& + \frac{\mu^*}{\sigma} \left(\frac{\partial^2 T^{*'}}{\partial x^{*2}} + \frac{\partial^2 T^{*'}}{\partial y^{*2}} + \frac{\partial^2 T^{*'}}{\partial z^{*2}} \right) + \frac{1}{\sigma} \left(\frac{\partial \mu^*}{\partial y^*} \right) \left(\frac{\partial T^{*'}}{\partial y^*} \right) \\
& + \frac{\mu^{*'}}{\sigma} \frac{\partial^2 T^*}{\partial y^{*2}} + \frac{1}{\sigma} \left(\frac{\partial \mu^{*'}}{\partial y^*} \right) \left(\frac{\partial T^*}{\partial y^*} \right) \\
& + \frac{2\mu^*}{c_p} \left[\frac{\partial u^*}{\partial y^*} \left(\frac{\partial u^{*'}}{\partial y^*} + \frac{\partial v^{*'}}{\partial x^*} \right) + \frac{\partial w^*}{\partial y^*} \left(\frac{\partial w^{*'}}{\partial y^*} + \frac{\partial v^{*'}}{\partial z^*} \right) \right] \\
& + \frac{\mu^{*'}}{c_p} \left[\left(\frac{\partial \mu^*}{\partial y^*} \right)^2 + \left(\frac{\partial w^*}{\partial y^*} \right)^2 \right] \quad (B5)
\end{aligned}$$

State:

$$\frac{p^{*'}}{p^*} = \frac{\rho^{*'}}{\rho^*} + \frac{T^{*'}}{T^*} \quad (B6)$$

Equations (B1) to (B6) are made dimensionless as in the text of this report. A disturbance of the form of equation (15) is introduced and the subsequent transformations carried out. The resulting transformed equations are:

Continuity:

$$i(W - c)r + \rho' \varphi = -\rho(\varphi' + i\mathcal{F}) \quad (B7)$$

Momentum:

$$\begin{aligned} \rho[i(W - c)\mathcal{F} + W'\varphi] = & -\frac{i\pi}{\gamma M_{\text{ref}}^2} + \frac{\mu}{\alpha \text{Re}_{\text{ref}}} [\mathcal{F}'' + \alpha^2(i\varphi' - 2\mathcal{F})] \\ & + \frac{2}{3} \frac{\mu_2 - \mu}{\alpha \text{Re}_{\text{ref}}} \alpha^2(i\varphi' - \mathcal{F}) + \frac{1}{\alpha \text{Re}_{\text{ref}}} [mW'' + m'W' + \mu'(\mathcal{F}' + i\alpha^2\varphi)] \end{aligned} \quad (\text{B8})$$

$$\begin{aligned} i\rho\alpha^2(W - c)\varphi = & -\frac{\pi'}{\gamma M_{\text{ref}}^2} + \frac{\mu}{\text{Re}_{\text{ref}}} \alpha(2\varphi'' + i\mathcal{F}' - \alpha^2\varphi) \\ & + \frac{2\alpha}{3} \frac{\mu_2 - \mu}{\text{Re}_{\text{ref}}} (\varphi'' + i\mathcal{F}') + \frac{\alpha}{\text{Re}_{\text{ref}}} \left[imW' + 2\mu'\varphi' + \frac{2}{3} (\mu_2' - \mu')(\varphi' + i\mathcal{F}') \right] \end{aligned} \quad (\text{B9})$$

Energy:

$$\begin{aligned} \rho[i(W - c)\theta + T'\varphi] = & i\left(\frac{\gamma - 1}{\gamma}\right)(W - c)\pi + \frac{1}{\alpha \text{Re}_{\text{ref}}} [\mu(\theta'' - \alpha^2\theta) + (mT')' + \mu'\theta'] \\ & + \frac{(\gamma - 1)M_{\text{ref}}^2}{\alpha \text{Re}_{\text{ref}}} \left(m \left\{ \left[\frac{\bar{u}' \cos \Psi}{\cos(\Theta - \Psi)} \right]^2 + \left[\frac{\bar{w}' \sin \Psi}{\cos(\Theta - \Psi)} \right]^2 \right\} \right. \\ & \left. + 2\mu \left[\frac{\bar{u}'f' \cos \Psi + \bar{w}'h' \sin \Psi}{\cos(\Theta - \Psi)} + i\alpha^2 W'\varphi \right] \right) \end{aligned} \quad (\text{B10})$$

State:

$$\pi = \frac{r}{\rho} + \frac{\theta}{T} \quad (\text{B11})$$

In these equations, as in the main text,

$$W = \frac{\bar{u} + \bar{w} \tan \Theta \tan \Psi}{1 + \tan \Theta \tan \Psi} \quad (\text{B12})$$

The fluctuating viscosity can be related to the temperature fluctuation through

$$m = \theta \left(\frac{d\mu}{dT} \right) \quad (\text{B13})$$

while the normal gradient of mean viscosity can be expressed

$$\mu' = T' \left(\frac{d\mu}{dT} \right) \quad (B14)$$

The continuity and momentum equations (B7) to (B9) are identical in form to those for two-dimensional boundary layers with respect to two-dimensional disturbances; however, in the energy equation the dissipation term having coefficient $(\gamma - 1)M_{\text{ref}}^2 / \alpha Re_{\text{ref}}$ does not transform as is seen by the inability to express this term in W and \mathcal{X} alone. Even in the case of two-dimensional boundary layers ($\Psi = 0$) with respect to three-dimensional disturbances, the dissipation term is

$$\frac{(\gamma - 1)M_{\text{ref}}^2}{\alpha Re_{\text{ref}}} \left[m \left(\frac{\bar{u}'}{\cos \Theta} \right)^2 + 2\mu \bar{u}' \left(\frac{f'}{\cos \Theta} + i\alpha^2 \phi \right) \right]$$

and does not transform confirming the result of Dunn and Lin (refs. 4 and 7).

The Dunn-Lin ordering procedure (eq. (47)) used to obtain the viscous solutions to disturbance equations indicates that the leading dissipation term

$$\frac{(\gamma - 1)M_{\text{ref}}^2}{\alpha Re_{\text{ref}}} 2\mu \left[\frac{\bar{u}' f' \cos \Psi + \bar{w}' h' \sin \Psi}{\cos(\Theta - \Psi)} \right]$$

is one order of magnitude smaller than the leading conduction term

$\frac{1}{\alpha Re_{\text{ref}}} \mu \theta''$; thus, if the analysis is restricted to consideration of the leading viscous terms only, the disturbance equations for three-dimensional boundary layers to three-dimensional disturbances transform to those for a two-dimensional boundary layer to two-dimensional disturbances. Such restriction is valid when $\frac{1}{(\alpha Re_{\text{ref}})^{1/2}} \ll 1$.

APPENDIX C

SECULAR EQUATION FOR LARGE z

In terms of $F(z)$ and $\tilde{G}(z_0)$ alone, the secular equation (82) is expressed as

$$-\frac{W_W^I}{c} G_W = \left(\frac{F + K\tilde{G}}{\frac{1}{1+\lambda} - F - \tilde{G} \frac{T_W^I c}{T_W W_W^I}} \right) \quad (C1)$$

The asymptotic variation of $F(z)$ from Lin (ref. 14) is

$$F(z) \approx \frac{1}{z^{3/2} e^{-i\pi/4} - 5/4} \quad (C2)$$

Let

$$p = \frac{z^{-3/2}}{\sqrt{2}} \quad (C3)$$

so that

$$z^{-3/2} e^{i\pi/4} = p + ip \quad (C4)$$

The asymptotic forms of $F(z)$ and $\tilde{G}(z_0)$ in terms of p to order p^2 are

$$F(z) \approx p + ip \left(1 + \frac{5}{2} p \right) \quad (C5)$$

and

$$\tilde{G}(z_0) = \frac{F(z_0)}{1 - F(z_0) - z_0 F'(z_0)} \approx \sigma^{-1/2} \left[p + ip \left(1 + \frac{3}{2} \sigma^{-1/2} p \right) \right] \quad (C6)$$

When (C5) and (C6) are substituted into equation (C1), the secular equation for large z (small p) is written

$$-\frac{W'_w}{c} G_w = (1 + \lambda)(1 + K\sigma^{-1/2})_p \left\{ 1 + i \left[1 + 2(1 + \lambda)_p \left(1 + \frac{T'_w c}{T_w W'_w} \sigma^{-1/2} \right) + \left(\frac{5 + 3K\sigma^{-1}}{2 + 2K\sigma^{-1/2}} \right)_p \right] + \dots \right\} \quad (C7)$$

Note in equation (C7) that to order p , the real and imaginary parts of the right side are equal. The quantity z is related to p by

$$z = \left(\frac{1}{p\sqrt{2}} \right)^{2/3} \quad (C8)$$

APPENDIX D

SECULAR EQUATION USING HEISENBERG-LIN INVISCID SOLUTION

In order to obtain the Heisenberg-Lin inviscid solutions, the inviscid equations (36) are written as a second-order differential equation in φ :

$$\frac{d}{dy} \left[\frac{(W - c)\varphi' - W'\varphi}{T - \tilde{M}^2(W - c)^2} \right] - \frac{\alpha^2(W - c)}{T} \varphi = 0 \quad (D1)$$

Then a solution of the form

$$\varphi = \varphi^{(0)} + \alpha^2 \varphi^{(1)} + \alpha^4 \varphi^{(2)} + \dots \quad (D2)$$

is sought. The two solutions that result from this procedure (following ref. 6) are

$$\varphi_1 = (W - c) \sum_{n=0}^{\infty} \alpha^{2n} h_{2n} \quad (D3a)$$

and

$$\varphi_2 = (W - c) \sum_{n=0}^{\infty} \alpha^{2n} k_{2n+1} \quad (D3b)$$

where

$$h_0 = 1; \quad h_{2n} = \int_0^y \frac{T - \tilde{M}^2(W - c)^2}{(W - c)^2} \int_0^y \frac{(W - c)^2}{T} h_{2n-2} dy dy \quad (D4)$$

and

$$\left. \begin{aligned} k_1 &= \int_0^y \frac{T - \tilde{M}^2(W - c)^2}{(W - c)^2} dy \\ k_{2n+1} &= \int_0^y \frac{T - \tilde{M}^2(W - c)^2}{(W - c)^2} \int_0^y \frac{(W - c)^2}{T} k_{2n-1} dy dy \end{aligned} \right\} \quad (D5)$$

Now the ratio $\phi_{\text{inv},w}/\mathcal{F}_{\text{inv},w}$ is found by first obtaining that combination of solutions 1 and 2 that, in the outer uniform mean flow, satisfy the outer condition $\phi_e' + \alpha \sqrt{1 - \tilde{M}^2(1 - c)^2} \phi_e = 0$, and then using the following identity from the inviscid equations

$$\mathcal{F}_{\text{inv},w} = 1 \frac{T_w \phi_{\text{inv},w}' + W_w' c \tilde{M}^2 \phi_{\text{inv},w}}{T_w - \tilde{M}^2 c^2}$$

The resulting expression is

$$\frac{\phi_{\text{inv},w}}{\mathcal{F}_{\text{inv},w}} = \frac{1}{\frac{W_w'}{c} + \frac{T_w}{c^2} \frac{\phi_{1,e}' + \alpha \sqrt{1 - \tilde{M}^2(1 - c)^2} \phi_{1,e}}{\phi_{2,e}' + \alpha \sqrt{1 - \tilde{M}^2(1 - c)^2} \phi_{2,e}}} \quad (D6)$$

The resulting secular equation is

$$(u - 1) + iv = \left[\frac{\Phi(z) - 1}{1 - \frac{\lambda}{\lambda + 1} \Phi(z)} \right] \left[\frac{1 + K \frac{\tilde{G}(z_0)}{F(z)}}{1 - \frac{\tilde{G}(z_0)\Phi(z)}{1 - \frac{\lambda}{\lambda + 1} \Phi(z)} \frac{T_w' c}{T_w W_w'}} \right] \quad (D7)$$

where u and v are given by

$$u = L + \frac{W_w' c}{T_w} \frac{\sqrt{1 - \tilde{M}^2(1 - c)^2}}{(1 - c)^2} \left(\frac{1}{\alpha} - A_1 + A_{2r} \alpha + \dots \right) \quad (D8a)$$

and

$$v = v_0 + \frac{W'_w c}{T_w} \frac{\sqrt{1 - \tilde{M}^2(1 - c)^2}}{(1 - c)^2} (A_{2i}^\alpha + \dots) \quad (D8b)$$

Further

$$L = 1 + \frac{W'_w c}{T_w} K_{1r} \quad (D9a)$$

and

$$\begin{aligned} v_0 &= \frac{W'_w c}{T_w} K_{1i} \\ &= -\pi \frac{W'_w}{T_w} \frac{c T_c^2}{W'_c} \left[\frac{d}{dy} \left(\frac{W'}{T} \right) \right]_c \end{aligned} \quad (D9b)$$

The coefficients A_1 and A_2 are given by

$$A_1 = \frac{\sqrt{1 - \tilde{M}^2(1 - c)^2}}{(1 - c)^2} H_1 \quad (D10a)$$

and

$$A_2 = A_1^2 - 2H_2 \quad (D10b)$$

and K_1 , H_1 , and H_2 by

$$K_1 = \int_0^\delta \frac{T - \tilde{M}^2(W - c)^2}{(W - c)^2} dy \quad (D11a)$$

$$H_1 = \int_0^\delta \frac{(W - c)^2}{T} dy \quad (D11b)$$

and

$$H_2 = \int_0^\delta \frac{T - \tilde{M}^2(W - c)^2}{(W - c)^2} \int_0^y \frac{(W - c)^2}{T} dy dy \quad (D11c)$$

Expressions for u and v to order α^3 as well as procedures for evaluating the H_n and K_n integrals are given by Mack (ref. 6).

The calculation of neutral stability characteristics is usually performed by the following iteration procedure: First, assume $\alpha = 0$ in equation (D8b), then find z for which $v_0(c) = \text{Im} [\text{RHS (eq. (D7))}]$. Set $u = 1 + R_1 [\text{RHS (eq. (D7))}]$ and solve for the first approximation to α by using equation (D8a).

$$\alpha_1 = \frac{W'_w c}{T_w (1 - c)^2} \frac{\sqrt{1 - \tilde{M}^2 (1 - c)^2}}{(u - L) + \frac{W'_w c}{T_w} \left[\frac{1 - \tilde{M}^2 (1 - c)^2}{(1 - c)^4} \right] H_1} \quad (D12)$$

The next iteration on α is carried out by substituting α_1 into equation (D8b) to give v_1 and the procedure is repeated with the A_{2r} term in equation (D8a). Successive iterations can be carried out with the additional terms of equation (D8) given by Mack (ref. 6). The value of $\alpha \tilde{Re}$ can be computed from the final value of z

$$\alpha \tilde{Re} = \frac{z^3}{\left(\frac{3}{2} \int_0^{y_c} \sqrt{\frac{c - W}{v}} dy \right)^2} \quad (87)$$

and then with α known the value of \tilde{Re} is obtained.

The procedure of this appendix, perhaps, has a slight advantage over that of Mack since the quantity λ appears only on the right side of the secular equation (D7) rather than on both sides.

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TABLE I. - TIETJENS AND AUXILIARY FUNCTIONS

[For $0 \leq z \leq 6.76$, the table is taken from Mack (ref. 6). For $6.80 \leq z \leq 8.00$, the functions F_r and F_1 are interpolated from the data of Miles (ref. 15), and the other functions are calculated from their definitions (83) and (84). For $8.10 < z < 10.00$, the functions F_r and F_1 are from Miles and the other functions are calculated from their definitions. It is expected that all numbers are correct to ± 0.00002 .]

| z | F_r | F_1 | Φ_r | Φ_1 | \bar{G}_r | \bar{G}_1 | z | F_r | F_1 | Φ_r | Φ_1 | \bar{G}_r | \bar{G}_1 |
|------|----------|----------|----------|----------|-------------|-------------|------|---------|----------|----------|----------|-------------|-------------|
| 0.04 | 16.98520 | -9.70547 | -0.28788 | -0.17479 | 20.64871 | -11.73373 | 2.24 | 0.57119 | -0.01315 | 2.32982 | -0.07144 | 0.55086 | 0.37636 |
| 0.08 | 8.58163 | -4.65142 | -0.09358 | -0.05988 | 10.48902 | -5.86208 | 2.28 | .56644 | -.00397 | 2.30629 | -.02112 | .52294 | .38167 |
| .12 | 5.78146 | -3.23248 | -.14354 | -.09704 | 7.10585 | -3.90332 | 2.32 | .56172 | .00527 | 2.28132 | .02741 | .48545 | .38539 |
| .16 | 4.38231 | -2.42257 | -.19541 | -.13996 | 5.41699 | -2.92183 | 2.36 | .55701 | .01457 | 2.25495 | .07416 | .46553 | .38757 |
| .20 | 3.54347 | -1.93612 | -.24893 | -.18949 | 4.40567 | -2.33144 | 2.40 | .55229 | .02394 | 2.22720 | .11910 | .44227 | .38831 |
| .24 | 2.98484 | -1.61148 | -.30366 | -.24654 | 3.73341 | -1.93677 | 2.44 | .54753 | .03339 | 2.19811 | .16221 | .41678 | .38770 |
| .28 | 2.58634 | -1.37917 | -.35901 | -.31213 | 3.25416 | -1.65308 | 2.48 | .54271 | .04292 | 2.16771 | .20545 | .39215 | .38582 |
| .32 | 2.28794 | -1.20460 | -.41415 | -.38735 | 2.89670 | -1.43942 | 2.52 | .53782 | .05253 | 2.13607 | .24279 | .36843 | .38278 |
| .36 | 2.05823 | -1.06849 | -.46792 | -.47335 | 2.61980 | -1.27192 | 2.56 | .53283 | .06223 | 2.10323 | .28016 | .34570 | .37870 |
| .40 | 1.87127 | -.95934 | -.51878 | -.57122 | 2.39924 | -1.13664 | 2.60 | .52772 | .07201 | 2.06927 | .31552 | .32398 | .37367 |
| .44 | 1.72027 | -.86972 | -.56483 | -.68203 | 2.22005 | -1.02502 | 2.64 | .52246 | .08189 | 2.03426 | .34882 | .30331 | .36781 |
| .48 | 1.59475 | -.79477 | -.60357 | -.80855 | 2.07170 | -.93069 | 2.68 | .51704 | .09184 | 1.99829 | .38000 | .28371 | .36122 |
| .52 | 1.48886 | -.73106 | -.63204 | -.94520 | 1.94710 | -.84978 | 2.72 | .51142 | .10188 | 1.96146 | .40901 | .26518 | .35401 |
| .56 | 1.39837 | -.67621 | -.64674 | -1.09782 | 1.84105 | -.77927 | 2.76 | .50559 | .11200 | 1.92388 | .43582 | .24771 | .34626 |
| .60 | 1.32021 | -.62841 | -.64372 | -1.26330 | 1.75001 | -.71696 | 2.80 | .49951 | .11219 | 1.88566 | .46037 | .23130 | .33807 |
| .64 | 1.25208 | -.58634 | -.61884 | -1.43945 | 1.67093 | -.66130 | 2.84 | .49317 | .13245 | 1.84692 | .48265 | .21592 | .32953 |
| .68 | 1.19220 | -.54895 | -.56816 | -1.62274 | 1.60200 | -.61097 | 2.88 | .48653 | .14276 | 1.80790 | .50263 | .20154 | .32071 |
| .72 | 1.13920 | -.51548 | -.48827 | -1.80810 | 1.54125 | -.56497 | 2.92 | .47958 | .15312 | 1.76843 | .52030 | .18814 | .31168 |
| .76 | 1.09200 | -.48527 | -.37714 | -1.98919 | 1.48743 | -.52259 | 2.96 | .47227 | .16350 | 1.72894 | .53566 | .17569 | .30252 |
| .80 | 1.04873 | -.45784 | -.23449 | -2.15869 | 1.43954 | -.48317 | 3.00 | .46458 | .17390 | 1.68948 | .54873 | .16414 | .29327 |
| .84 | 1.01168 | -.43277 | -.06233 | -2.30900 | 1.39658 | -.44622 | 3.04 | .45649 | .18429 | 1.65018 | .55953 | .15346 | .28400 |
| .88 | .97729 | -.40975 | .13486 | -2.43307 | 1.35794 | -.41127 | 3.08 | .44797 | .19464 | 1.61119 | .56809 | .14360 | .27474 |
| .92 | .94607 | -.38846 | .35065 | -2.52556 | 1.32289 | -.37798 | 3.12 | .43898 | .20493 | 1.57264 | .57446 | .13453 | .26555 |
| .96 | .91762 | -.36871 | .57718 | -2.58320 | 1.29104 | -.34611 | 3.16 | .42951 | .21513 | 1.53465 | .57871 | .12620 | .25645 |
| 1.00 | .89162 | -.35029 | .80612 | -2.60534 | 1.26190 | -.31541 | 3.20 | .41952 | .22519 | 1.49736 | .58090 | .11857 | .24747 |
| 1.04 | .86778 | -.33303 | 1.02983 | -2.59383 | 1.23504 | -.28558 | 3.24 | .40900 | .23509 | 1.46069 | .58111 | .11161 | .23866 |
| 1.08 | .84585 | -.31680 | 1.24187 | -2.55230 | 1.21019 | -.25851 | 3.28 | .39791 | .24476 | 1.42534 | .57943 | .10527 | .23002 |
| 1.12 | .82565 | -.30148 | 1.43751 | -2.48566 | 1.18696 | -.22805 | 3.32 | .38625 | .25417 | 1.39082 | .57596 | .09851 | .22158 |
| 1.16 | .80698 | -.28695 | 1.61397 | -2.39928 | 1.16514 | -.20007 | 3.36 | .37400 | .26324 | 1.35741 | .57081 | .09430 | .21336 |
| 1.20 | .78968 | -.27313 | 1.76988 | -2.29843 | 1.14446 | -.17245 | 3.40 | .36115 | .27193 | 1.32521 | .56408 | .08960 | .20536 |
| 1.24 | .77363 | -.25993 | 1.90532 | -2.18785 | 1.12471 | -.14512 | 3.44 | .34769 | .28016 | 1.29427 | .55588 | .08538 | .19761 |
| 1.28 | .75871 | -.24730 | 2.02123 | -2.07154 | 1.10568 | -.11802 | 3.48 | .33364 | .28787 | 1.26467 | .54633 | .08160 | .19011 |
| 1.32 | .74480 | -.23516 | 2.11911 | -1.95271 | 1.08715 | -.09107 | 3.52 | .31900 | .29497 | 1.23645 | .53556 | .07823 | .18286 |
| 1.36 | .73192 | -.22347 | 2.20078 | -1.83364 | 1.06892 | -.06428 | 3.56 | .30380 | .30139 | 1.20966 | .52367 | .07524 | .17588 |
| 1.40 | .71969 | -.21216 | 2.26814 | -1.71667 | 1.05084 | -.03759 | 3.60 | .28807 | .30706 | 1.18432 | .51080 | .07260 | .16916 |
| 1.44 | .70832 | -.20120 | 2.32304 | -1.60244 | 1.03274 | -.01103 | 3.64 | .27186 | .31188 | 1.16046 | .49705 | .07028 | .16271 |
| 1.48 | .69765 | -.19055 | 2.36720 | -1.49191 | 1.01448 | .01541 | 3.68 | .25522 | .31579 | 1.13808 | .48255 | .06825 | .15652 |
| 1.52 | .68763 | -.18018 | 2.40215 | -1.38558 | .99590 | .04163 | 3.72 | .23824 | .31871 | 1.11719 | .46741 | .06650 | .15060 |
| 1.56 | .67820 | -.17004 | 2.42925 | -1.28360 | .97689 | .06760 | 3.76 | .22098 | .32057 | 1.09778 | .45173 | .06499 | .14495 |
| 1.60 | .66930 | -.16011 | 2.44968 | -1.18603 | .95733 | .09327 | 3.80 | .20356 | .32131 | 1.07983 | .43564 | .06370 | .13956 |
| 1.64 | .66090 | -.15037 | 2.46440 | -1.09278 | .93714 | .11953 | 3.84 | .18607 | .32089 | 1.06333 | .41922 | .06261 | .13442 |
| 1.68 | .65294 | -.14079 | 2.47425 | -1.00366 | .91622 | .14331 | 3.88 | .16865 | .31928 | 1.04825 | .40258 | .06170 | .12954 |
| 1.72 | .64540 | -.13133 | 2.47993 | -.91848 | .89451 | .16749 | 3.92 | .15142 | .31646 | 1.03456 | .38581 | .06096 | .12491 |
| 1.76 | .63824 | -.12200 | 2.48199 | -.83700 | .87196 | .19096 | 3.96 | .13450 | .31243 | 1.02221 | .36900 | .06035 | .12052 |
| 1.80 | .63142 | -.11278 | 2.48090 | -.75897 | .84859 | .21360 | 4.00 | .11805 | .30721 | 1.01116 | .35222 | .05987 | .11636 |
| 1.84 | .62491 | -.10360 | 2.47706 | -.68415 | .82435 | .23527 | 4.04 | .10218 | .30085 | 1.00136 | .33556 | .05951 | .11243 |
| 1.88 | .61868 | -.09450 | 2.47074 | -.61250 | .79931 | .25586 | 4.08 | .08703 | .29343 | .99277 | .31908 | .05923 | .10872 |
| 1.92 | .61271 | -.08545 | 2.46220 | -.54322 | .77349 | .27524 | 4.12 | .07272 | .28501 | .98554 | .30285 | .05904 | .10522 |
| 1.96 | .60697 | -.07642 | 2.45163 | -.47671 | .74697 | .29330 | 4.16 | .05934 | .27569 | .97899 | .28693 | .05892 | .10193 |
| 2.00 | .60143 | -.06742 | 2.43917 | -.41260 | .71984 | .30995 | 4.20 | .04699 | .26560 | .97368 | .27136 | .05885 | .09863 |
| 2.04 | .59607 | -.05842 | 2.42495 | -.35074 | .69220 | .32509 | 4.24 | .03574 | .25486 | .96935 | .25621 | .05883 | .09593 |
| 2.08 | .59087 | -.04942 | 2.40905 | -.29099 | .66417 | .33865 | 4.28 | .02564 | .24361 | .96594 | .24150 | .05885 | .09320 |
| 2.12 | .58580 | -.04040 | 2.39154 | -.23326 | .63588 | .35058 | 4.32 | .01671 | .23197 | .96338 | .22728 | .05889 | .09064 |
| 2.16 | .58085 | -.03136 | 2.37248 | -.17747 | .60747 | .36085 | 4.36 | .00898 | .22010 | .96161 | .21357 | .05896 | .08824 |
| 2.20 | .57598 | -.02227 | 2.35189 | -.12355 | .57908 | .36945 | 4.40 | .00259 | .20815 | .96058 | .20040 | .05903 | .08600 |

TABLE I. - Concluded. TIETJENS AND AUXILIARY FUNCTIONS

[For $0 < z < 6.76$, the table is taken from Mack (ref. 6). For $6.80 < z < 8.00$, the functions F_r and F_i are interpolated from the data of Miles (ref. 15), and the other functions are calculated from their definitions (83) and (84). For $8.10 < z < 10.00$, the functions F_r and F_i are from Miles and the other functions are calculated from their definitions. It is expected that all numbers are correct to ± 0.00002 .]

| z | F_r | F_i | ϕ_r | ϕ_i | \bar{G}_r | \bar{G}_i | z | F_r | F_i | ϕ_r | ϕ_i | \bar{G}_r | \bar{G}_i |
|------|----------|---------|----------|----------|-------------|-------------|-------|---------|---------|----------|----------|-------------|-------------|
| 4.44 | -0.00306 | 0.19617 | 0.96023 | 0.16780 | 0.05911 | 0.08391 | 6.64 | 0.04442 | 0.04497 | 1.04417 | 0.04913 | 0.04119 | 0.04427 |
| 4.48 | -0.00740 | .18436 | .96048 | .17577 | .05920 | .08195 | 6.68 | .04387 | .04487 | 1.04359 | .04897 | .04080 | .04395 |
| 4.52 | -0.01072 | .17279 | .96130 | .16434 | .05927 | .08012 | 6.72 | .04332 | .04474 | 1.04300 | .04878 | .04042 | .04344 |
| 4.56 | -0.01308 | .16155 | .96261 | .15350 | .05933 | .07841 | 6.76 | .04277 | .04458 | 1.04242 | .04855 | .04005 | .04303 |
| 4.60 | -0.01455 | .15072 | .96437 | .14327 | .05938 | .07681 | 6.80 | .04222 | .04440 | 1.04184 | .04830 | .03968 | .04262 |
| 4.64 | -.01523 | .14037 | .96653 | .13354 | .05942 | .07532 | 6.84 | .04167 | .04419 | 1.04127 | .04801 | .03932 | .04222 |
| 4.68 | -.01519 | .13054 | .96902 | .12461 | .05943 | .07393 | 6.88 | .04113 | .04396 | 1.04071 | .04771 | .03897 | .04183 |
| 4.72 | -.01452 | .12128 | .97180 | .11617 | .05942 | .07263 | 6.92 | .04059 | .04370 | 1.04015 | .04738 | .03862 | .04144 |
| 4.76 | -.01331 | .11259 | .97483 | .10832 | .05939 | .07141 | 6.96 | .04005 | .04343 | 1.03959 | .04703 | .03828 | .04106 |
| 4.80 | -.01164 | .10451 | .97806 | .10104 | .05933 | .07027 | 7.00 | .03951 | .04314 | 1.03904 | .04667 | .03795 | .04068 |
| 4.84 | -.00959 | .09702 | .98144 | .09432 | .05925 | .06920 | 7.04 | .03898 | .04282 | 1.03850 | .04628 | .03762 | .04031 |
| 4.88 | -.00722 | .09013 | .98495 | .08814 | .05913 | .06819 | 7.08 | .03847 | .04250 | 1.03798 | .04589 | .03729 | .03994 |
| 4.92 | -.00460 | .08382 | .98853 | .08246 | .05900 | .06725 | 7.12 | .03799 | .04216 | 1.03749 | .04547 | .03697 | .03957 |
| 4.96 | -.00180 | .07808 | .99217 | .07733 | .05883 | .06636 | 7.16 | .03752 | .04182 | 1.03703 | .04505 | .03666 | .03922 |
| 5.00 | .00113 | .07289 | .99583 | .07267 | .05864 | .06552 | 7.20 | .03706 | .04146 | 1.03656 | .04463 | .03635 | .03887 |
| 5.04 | .00414 | .06822 | .99947 | .06846 | .05842 | .06472 | 7.24 | .03661 | .04110 | 1.03612 | .04420 | .03605 | .03852 |
| 5.08 | .00720 | .06404 | 1.00308 | .06470 | .05817 | .06397 | 7.28 | .03617 | .04073 | 1.03568 | .04376 | .03575 | .03817 |
| 5.12 | .01026 | .06032 | 1.00662 | .06135 | .05790 | .06325 | 7.32 | .03575 | .04035 | 1.03526 | .04332 | .03546 | .03784 |
| 5.16 | .01328 | .05704 | 1.01009 | .05839 | .05761 | .06256 | 7.36 | .03536 | .03997 | 1.03488 | .04288 | .03517 | .03751 |
| 5.20 | .01626 | .05416 | 1.01345 | .05580 | .05729 | .06190 | 7.40 | .03497 | .03959 | 1.03450 | .04244 | .03489 | .03718 |
| 5.24 | .01915 | .05166 | 1.01670 | .05355 | .05695 | .06127 | 7.44 | .03460 | .03921 | 1.03413 | .04200 | .03461 | .03686 |
| 5.28 | .02194 | .04951 | 1.01992 | .05162 | .05660 | .06066 | 7.48 | .03425 | .03883 | 1.03379 | .04156 | .03434 | .03654 |
| 5.32 | .02462 | .04766 | 1.02290 | .04998 | .05622 | .06007 | 7.52 | .03390 | .03845 | 1.03345 | .04112 | .03407 | .03623 |
| 5.36 | .02717 | .04611 | 1.02562 | .04861 | .05582 | .05949 | 7.56 | .03357 | .03807 | 1.03313 | .04069 | .03380 | .03592 |
| 5.40 | .02958 | .04482 | 1.02829 | .04749 | .05541 | .05894 | 7.60 | .03324 | .03769 | 1.03281 | .04026 | .03354 | .03562 |
| 5.44 | .03185 | .04376 | 1.03079 | .04659 | .05499 | .05839 | 7.64 | .03293 | .03731 | 1.03251 | .03983 | .03328 | .03532 |
| 5.48 | .03397 | .04292 | 1.03313 | .04590 | .05455 | .05786 | 7.68 | .03263 | .03693 | 1.03223 | .03942 | .03302 | .03502 |
| 5.52 | .03594 | .04226 | 1.03529 | .04538 | .05410 | .05734 | 7.72 | .03235 | .03657 | 1.03196 | .03901 | .03277 | .03473 |
| 5.56 | .03776 | .04177 | 1.03729 | .04503 | .05365 | .05682 | 7.76 | .03207 | .03621 | 1.03169 | .03861 | .03252 | .03445 |
| 5.60 | .03942 | .04143 | 1.03911 | .04482 | .05318 | .05631 | 7.80 | .03180 | .03587 | 1.03143 | .03822 | .03227 | .03417 |
| 5.64 | .04093 | .04121 | 1.04076 | .04472 | .05270 | .05581 | 7.84 | .03154 | .03553 | 1.03118 | .03783 | .03203 | .03389 |
| 5.68 | .04230 | .04111 | 1.04224 | .04474 | .05222 | .05531 | 7.88 | .03129 | .03519 | 1.03094 | .03744 | .03179 | .03362 |
| 5.72 | .04351 | .04110 | 1.04357 | .04484 | .05174 | .05482 | 7.92 | .03104 | .03486 | 1.03070 | .03706 | .03155 | .03335 |
| 5.76 | .04459 | .04117 | 1.04473 | .04502 | .05125 | .05433 | 7.96 | .03081 | .03453 | 1.03048 | .03670 | .03132 | .03308 |
| 5.80 | .04553 | .04130 | 1.04574 | .04525 | .05076 | .05385 | 8.00 | .03058 | .03421 | 1.03026 | .03635 | .03109 | .03282 |
| 5.84 | .04633 | .04149 | 1.04660 | .04553 | .05026 | .05337 | 8.10 | .03004 | .03343 | 1.02975 | .03549 | .03053 | .03218 |
| 5.88 | .04701 | .04172 | 1.04733 | .04585 | .04977 | .05289 | 8.20 | .02952 | .03270 | 1.02925 | .03468 | .02998 | .03156 |
| 5.92 | .04757 | .04198 | 1.04791 | .04618 | .04928 | .05241 | 8.30 | .02903 | .03201 | 1.02878 | .03392 | .02949 | .03097 |
| 5.96 | .04802 | .04226 | 1.04838 | .04653 | .04879 | .05194 | 8.40 | .02855 | .03135 | 1.02832 | .03319 | .02892 | .03039 |
| 6.00 | .04836 | .04255 | 1.04872 | .04689 | .04830 | .05147 | 8.50 | .02810 | .03073 | 1.02788 | .03250 | .02842 | .02983 |
| 6.04 | .04860 | .04285 | 1.04895 | .04724 | .04781 | .05100 | 8.60 | .02765 | .03014 | 1.02745 | .03185 | .02794 | .02929 |
| 6.08 | .04874 | .04314 | 1.04908 | .04758 | .04733 | .05053 | 8.70 | .02721 | .02957 | 1.02702 | .03122 | .02748 | .02876 |
| 6.12 | .04880 | .04343 | 1.04912 | .04790 | .04685 | .05006 | 8.80 | .02678 | .02903 | 1.02660 | .03062 | .02695 | .02825 |
| 6.16 | .04878 | .04371 | 1.04906 | .04820 | .04638 | .04960 | 8.90 | .02636 | .02851 | 1.02619 | .03005 | .02654 | .02776 |
| 6.20 | .04868 | .04397 | 1.04893 | .04848 | .04591 | .04914 | 9.00 | .02595 | .02801 | 1.02579 | .02950 | .02611 | .02728 |
| 6.24 | .04851 | .04421 | 1.04872 | .04872 | .04545 | .04866 | 9.10 | .02554 | .02752 | 1.02539 | .02896 | .02568 | .02681 |
| 6.28 | .04828 | .04442 | 1.04845 | .04893 | .04499 | .04822 | 9.20 | .02514 | .02705 | 1.02500 | .02844 | .02527 | .02636 |
| 6.32 | .04800 | .04460 | 1.04812 | .04911 | .04454 | .04777 | 9.30 | .02475 | .02659 | 1.02461 | .02794 | .02486 | .02592 |
| 6.36 | .04767 | .04476 | 1.04774 | .04925 | .04410 | .04732 | 9.40 | .02436 | .02615 | 1.02423 | .02745 | .02446 | .02549 |
| 6.40 | .04730 | .04489 | 1.04732 | .04935 | .04366 | .04687 | 9.50 | .02399 | .02572 | 1.02387 | .02698 | .02408 | .02507 |
| 6.44 | .04688 | .04498 | 1.04686 | .04941 | .04323 | .04643 | 9.60 | .02362 | .02530 | 1.02350 | .02652 | .02371 | .02467 |
| 6.48 | .04643 | .04504 | 1.04636 | .04943 | .04281 | .04599 | 9.70 | .02326 | .02488 | 1.02315 | .02607 | .02335 | .02428 |
| 6.52 | .04596 | .04507 | 1.04584 | .04941 | .04240 | .04556 | 9.80 | .02291 | .02448 | 1.02281 | .02563 | .02300 | .02389 |
| 6.56 | .04546 | .04507 | 1.04530 | .04936 | .04199 | .04512 | 9.90 | .02257 | .02409 | 1.02247 | .02520 | .02265 | .02351 |
| 6.60 | .04495 | .04503 | 1.04474 | .04926 | .04159 | .04469 | 10.00 | .02223 | .02371 | 1.02213 | .02479 | .02231 | .02315 |

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| <p>NASA TN D-1220 National Aeronautics and Space Administration. STABILITY OF THREE-DIMENSIONAL COMPRESSIBLE BOUNDARY LAYERS. Eli Reshotko. June 1962. 43p. OTS price, \$1.25. (NASA TECHNICAL NOTE D-1220)</p> <p>For Reynolds numbers sufficiently large that the dissipation terms in the disturbance energy equation are negligible, the stability of a three-dimensional boundary layer to a plane-wave disturbance of arbitrary orientation reduces to a two-dimensional stability problem governed by the boundary-layer velocity profile in the direction of wave propagation and by the mean temperature profile. Solution procedures are presented and the eigenvalue problem formulated including temperature fluctuations and a thermal boundary condition.</p> | <p>I. Reshotko, Eli II. NASA TN D-1220</p> <p>(Initial NASA distribution: 1, Aerodynamics, aircraft; 2, Aerodynamics, missiles and space vehicles; 20, Fluid mechanics.)</p> | NASA |
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